WEP
Work, Energy, Power
What is Work?
What is Work?

• A force accelerates an object in a given direction
• Work gives you a measure of how efficient the force is to obtain its effect in the direction of the motion
• Work=Force*displacement in the direction of the force

\[ dW = \vec{F} \cdot d\vec{r} = F^T dr = \begin{bmatrix} F_x & F_y & F_z \end{bmatrix} \begin{bmatrix} dr_x \\ dr_y \\ dr_z \end{bmatrix} \]
Total work done!

\[ W_{1\to2} = \int_1^2 dW = \int_1^2 F_t \, ds = \]

\[ = \int_1^2 m \ddot{s} \, ds = m \int_1^2 \frac{\dot{s}}{dt} \, dt \, ds = \]

\[ = m \int_1^2 \dot{s} \, ds = m \left( \frac{v_2^2}{2} - \frac{v_1^2}{2} \right) \]
Kinetic Energy

\[ K_2 = K_1 + W_{1\rightarrow 2} \]

Note that

Kinetic energy: \[ \frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2 = \int_1^2 \vec{F} \cdot d\vec{r} \]

Linear momentum: \[ m\vec{v}_2 - m\vec{v}_1 = \int_1^2 \vec{F} \, dt \]
Problem 4.1.11

\( F = 10 \text{ N}, \ m = 0.45 \text{ kg}, \ \theta = 45 \text{ deg} \)

What is the speed at \( \theta = 90 \text{ deg} \)?

**GOAL**: Determine the launch speed of a payload from a catapult.

**GIVEN**: Force acting along the direction of travel, size of the catapult and the initial and final angle of the catapult arm.
\[
\begin{align*}
\dot{e}_r &= -\hat{e}_i \cos \theta + \hat{e}_j \sin \theta \\
\dot{e}_\theta &= \hat{e}_i \sin \theta + \hat{e}_j \cos \theta
\end{align*}
\]
FORMULATE EQUATIONS: The FBD gives the forces on the mass as

\[ F_\theta \, \ddot{e}_\theta + N \, \ddot{e}_r - mg \ddot{f} \]

To solve the problem we’ll use the energy/work formula:

\[ \frac{1}{2} m v_1^2 + W_{1-2} = \frac{1}{2} m v_2^2 \]

SOLVE:
First we need the total force \( F \) that acts along the direction of travel:

\[ F = F_\theta - mg \cos \theta \]

Our work/energy expression becomes

\[ 0 + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (F_\theta - mg \cos \theta) r d\theta = \frac{1}{2} m v_2^2 \]

\[ F_\theta r \frac{\pi}{4} - mgr (\sin \frac{\pi}{2} - \sin \frac{\pi}{4}) = \frac{1}{2} m v_2^2 \]

\[ (10 \text{ N})(1.5 \text{ m}) \frac{\pi}{4} - (0.45 \text{ kg})(9.81 \text{ m/s}^2)(1.5 \text{ m})(\sin \frac{\pi}{2} - \sin \frac{\pi}{4}) = \frac{1}{2} (0.45 \text{ kg}) v_2^2 \]

\[ v_2 = 6.61 \text{ m/s} \]
Kinetic and potential energy of the particles and conservative systems
Conservative forces
Conservative forces

- **Definition:** the work done in moving an object from one place to another is **COMPLETELY** independent of the path taken.
Conservative forces

- Only the endpoints of the path matter!
- Conservative forces can always be expressed as derivatives (gradients) of potential functions.
Conservative forces

\[ F_c = -\frac{dV}{ds} \quad \rightarrow \quad V_2 - V_1 = -\int F_c \, ds \]

Potential of spring, gravitational field?
Mixed forces

\[ W_{1\to2} = \int_{s_1}^{s_2} (F_c + F_{nc}) \, ds = \int_{s_1}^{s_2} F_c \, ds + \int_{s_1}^{s_2} F_{nc} \, ds = \]

\[ = V_1 - V_2 + \int_{s_1}^{s_2} F_{nc} \, ds = V_1 - V_2 + W_{nc \, 1\to2} \]

recall: \( K_2 = K_1 + W_{1\to2} \)

Therefore

\[ K_2 - K_1 = V_1 - V_2 + W_{nc \, 1\to2} \]
Problem 4.2.8

**GOAL:** Determine the impact speed of a mass against the ceiling.

Spring un-stretched = 1 m.

**GIVEN:** Initial configuration of the system and spring constants.

10 kg

80 N/m

4 m

2 m

2 m
**FORMULATE EQUATIONS:**
We will employ conservation of energy from State 1 to State 2:
\[ \kappa \mathcal{E}_1 + \mathcal{P} \varepsilon_1 = \kappa \mathcal{E}_2 + \mathcal{P} \varepsilon_2 \]

**SOLVE:**
Initially (State 1) the kinetic energy is zero and the springs are stretched by an amount \(\sqrt{4^2 + 2^2} \text{ m} - 1 \text{ m}\) and at State 2 are stretched \((2 \text{ m} - 1 \text{ m})\). Our energy conservation equation is
\[
2 \frac{1}{2} k \left( \sqrt{4^2 + 2^2} \text{ m} - 1 \text{ m} \right)^2 = \frac{1}{2} mv^2 + 2 \frac{1}{2} k (2 \text{ m} - 1 \text{ m})^2 + mg(4 \text{ m})
\]

\[
2 \frac{1}{2} (80 \text{ N/m}) (3.472 \text{ m})^2 = \frac{1}{2} (10 \text{ kg}) v^2 + 2 \frac{1}{2} (80 \text{ N/m})(1 \text{ m})^2 + (10 \text{ kg})(9.81 \text{ m/s}^2)(4 \text{ m})
\]

\[
v = 9.92 \text{ m/s}
\]
Watt is Power?

- **The rate of work is the power**: power is how much work can be done per unit time.
- Consider lifting 10 kg to 1 m. This can be done in 10 s or in 1 s ... with 10 times more power!

\[
\text{Translational: } P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}
\]

\[
\text{Rotational: } P = \frac{dW}{dt} = T \frac{d\varphi}{dt} = F \omega
\]

**Watt (W):** 1 W = 1 J/s = 1 N \cdot m/s

**horsepower (hp):** 1 hp = 550 ft \cdot lb/s
\[ P = \vec{F} \cdot \vec{v} = F_t \, v = m a_t \, v = m \frac{dv}{dt} \]

multiplying by \( dt \) and dividing by \( P \)

\[
dt = \frac{m}{P} \, v \, dv \quad \int_{t_1}^{t_2} dt = \frac{m}{P} \int_{v_1}^{v_2} v \, dv \quad \rightarrow \quad t_2 - t_1 = \frac{m}{P} \left( \frac{v_2^2}{2} - \frac{v_1^2}{2} \right)
\]

\[ a_t = \frac{dv}{dt} \quad \rightarrow \quad a_t \, \frac{dx}{dt} = v \, \frac{dv}{dt} \quad \rightarrow \quad a_t \, dx = v \, dv \quad \rightarrow \quad P = ma_t \, v = m \frac{v^2 \, dv}{dx} \]

multiplying by \( dx \) and dividing by \( P \)

\[
dx = \frac{m}{P} \, v^2 \, dv \quad \rightarrow \quad \int_{x_1}^{x_2} dx = \frac{m}{P} \int_{v_1}^{v_2} v^2 \, dv \quad \rightarrow \quad x_2 - x_1 = \frac{m}{P} \left( \frac{v_2^3}{3} - \frac{v_1^3}{3} \right)
\]
GOAL: Find the percentage increase in power output for double the speed.

GIVEN: Relation between aerodynamic drag and speed.

Aerodynamic force increases as $F_d = av^2$ where $a = a$(cross section, fluid density)
**GOAL:** Find the percentage increase in power output for double the speed.

**GIVEN:** Relation between aerodynamic drag and speed.

Aerodynamic force increases as $F_d = av^2$ where $a = a$ (cross section, fluid density)

**ASSUME:** Let’s assume that, for the two cases in which we’re interested, the force exerted by the body exactly matches the drag force, such that it is not accelerating.

**FORMULATE EQUATIONS:**

Drag:

$$F_d = av^2$$  \hspace{1cm} (1)

Power:

$$P = F_d v$$  \hspace{1cm} (2)

**SOLVE:** Let $v_1$ be the initial velocity, and $v_2$ be double the initial velocity: $v_2 = 2v_1$. The expressions for power at these two states are

$$P_1 = F_d v_1 = av_1^3$$

$$P_2 = F_d v_2 = a(2v_1)^3$$

The percentage increase is given by

$$\% \text{ increase} = \frac{P_2 - P_1}{P_1} \times 100 = \frac{a(8v_1^3 - v_1^3)}{av_1^3} \times 100 = \frac{7av_1^3}{av_1^3} \times 100 = 700\%$$

$$\% \text{ increase} = 700\%$$
Figure 2.57  (p. 70)
Hovering Hummingbird.
Figure 2.58 (p. 70)
Hummingbird schematic.
Energy is conserved

\[ \ddot{a} \cdot \ddot{v} = \frac{1}{2} (\ddot{a} \cdot \ddot{v} + \dddot{v} \cdot \ddot{a}) = \frac{1}{2} \frac{d}{dt} (\dddot{v} \cdot \ddot{v}) = \frac{d}{dt} \left( \frac{1}{2} v^2 \right) \]

(Power) \[ \vec{F} \cdot \vec{v} = m \dddot{a} \cdot \ddot{v} = \frac{d}{dt} \left( \frac{1}{2} mv^2 \right) = \frac{dK}{dt} \]

Consider conservative force \( F \) is a function of only \( \vec{r} \)

\[ \vec{F} = \vec{F}(\vec{r}) \quad \text{and} \quad \vec{F} = -\nabla V \]

\( V \) is called the potential energy. Such forces are called conservative forces and systems for which the forces are conservative are called conservative systems.

If \( \vec{F} = -\nabla V \) then \( \vec{F} \cdot \vec{v} = -\nabla V \cdot \ddot{v} = -\frac{dV}{dt} \)

in fact: \[ \frac{dV}{dt} = \frac{\partial V}{\partial x} \frac{dx}{dt} + \frac{\partial V}{\partial y} \frac{dy}{dt} + \frac{\partial V}{\partial z} \frac{dz}{dt} = \nabla V \cdot \ddot{v} \]

\[ \frac{d}{dt} (K + V) = 0 \quad \rightarrow \quad K + V = \mathcal{E} = \text{constant} \]
Problem 4.2.19

GOAL: Find the distance the cyclist can coast.
GIVEN: \( v_1 = 25\text{mph}, \) 6\% grade, no air resistance or friction

FORMULATE EQUATIONS: The energy balance is given by

\[
\mathcal{KE}_2 + \mathcal{PE}_2 = \mathcal{KE}_1 + \mathcal{PE}_1
\]

(1)

SOLVE:
Rearranging (1) \(\Rightarrow\)

\[
\mathcal{PE}_2 - \mathcal{PE}_1 = \mathcal{KE}_1 - \mathcal{KE}_2
\]

(2)

Because the cyclist will be at rest at state 2, \( \mathcal{KE}_2 = 0 \). The change in potential energy in terms of \( d \) is given by

\[
\mathcal{PE}_2 - \mathcal{PE}_1 = mgd\sin \theta
\]

(3)

(3) \(\rightarrow\) (2) \(\Rightarrow\)

\[
mgd\sin \theta = \frac{1}{2}mv_1^2
\]

\[
d = \frac{v_1^2}{2g \sin \theta} = \frac{(25 \text{ mph} \times 1.4667 \frac{\text{ft/s}}{\text{mph}})^2}{2(32.2 \text{ ft/s}^2) \sin [\tan^{-1}(0.06)]} = 348.6 \text{ ft}
\]

\[
d = 348.6 \text{ ft}
\]