SAMPLE 2.39 Center of mass in 1-D: Three particles (point masses) of mass 2 kg, 3 kg, and 3 kg, are welded to a straight massless rod as shown in the figure. Find the location of the center-of-mass of the assembly.

Solution Let us select the first mass, $m_1 = 2$ kg, to be at the origin of our co-ordinate system with the $x$-axis along the rod. Since all the three masses lie on the $x$-axis, the center-of-mass will also lie on this axis. Let the center-of-mass be located at $x_{cm}$ on the $x$-axis. Then,

\[
m_{cm} x_{cm} = \sum_{i=1}^{3} m_i x_i = m_1 x_1 + m_2 x_2 + m_3 x_3
\]

\[
= m_1 (0) + m_2 (\ell) + m_3 (2\ell)
\]

\[
\Rightarrow x_{cm} = \frac{m_1 \ell + m_2 (2\ell)}{m_1 + m_2 + m_3}
\]

\[
= \frac{2 \text{ kg} \cdot 0.2 \text{ m} + 3 \text{ kg} \cdot 0.4 \text{ m}}{2 + 3 + 3} \text{ kg}
\]

\[
= \frac{1.8}{8} \text{ m} = 0.225 \text{ m}.
\]

Figure 2.83:

Alternatively, we could find the center-of-mass by first replacing the two 3 kg masses with a single 6 kg mass located in the middle of the two masses (the center-of-mass of the two equal masses) and then calculate the value of $x_{cm}$ for a two particle system consisting of the 2 kg mass and the 6 kg mass (see Fig. 2.84):

\[
x_{cm} = \frac{6 \text{ kg} \cdot 0.3 \text{ m}}{8 \text{ kg}} = \frac{1.8}{8} \text{ m} = 0.225 \text{ m}.
\]

Figure 2.84:

SAMPLE 2.40 Center of mass in 2-D: Two particles of mass $m_1 = 1$ kg and $m_2 = 2$ kg are located at coordinates $(1, 2)$ m and $(2, 5)$ m, respectively, in the $xy$-plane. Find the location of their center-of-mass.

Solution Let $\vec{r}_{cm}$ be the position vector of the center-of-mass. Then,

\[
m_{cm} = m_1 r_1 + m_2 r_2
\]

\[
\Rightarrow r_{cm} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}
\]

\[
= \frac{1 \text{ kg}(1 \text{ m} + 2 \text{ m}j) + 2 \text{ kg}(2 \text{ m} + 5 \text{ mj})}{3 \text{ kg}}
\]

\[
= \frac{(1 \text{ m} - 4 \text{ mj}) + (2 \text{ m}j + (0 \text{ m}j))}{3} = -1 \text{ m} + 4 \text{ mj}.
\]

Thus the center-of-mass is located at the coordinates $(1, 4, 4)$.

\[
(x_{cm}, y_{cm}) = (-1, 4, 4)
\]

Geometrically, this is just a 1-D problem like the previous sample. The center-of-mass has to be located on the straight line joining the two masses. Since the center-of-mass is a point about which the distribution of mass is balanced, it is easy to see (see Fig. 2.85) that the center-of-mass must lie one-third way from $m_2$ on the line joining the two masses so that $2 \text{ kg} \cdot (d/3) = 1 \text{ kg} \cdot (2d/3)$.  

Figure 2.85:
SAMPLE 2.41 Location of the center-of-mass. A structure is made up of three point masses, \( m_1 = 1 \text{ kg}, \ m_2 = 2 \text{ kg} \) and \( m_3 = 3 \text{ kg} \), connected rigidly by massless rods. At the moment of interest, the coordinates of the three masses are \((1.25 \text{ m}, 3 \text{ m}), (2 \text{ m}, 2 \text{ m}), \) and \((0.75 \text{ m}, 0.5 \text{ m})\), respectively. At the same instant, the velocities of the three masses are \(2 \text{ m/s} \hat{i}, \ 2 \text{ m/s}(\hat{i} - 1.5 \hat{j})\) and \(1 \text{ m/s} \hat{j}\), respectively. Find the coordinates of the center-of-mass of the structure.

**Solution** Just for fun, let us do this problem two ways — first using scalar equations for the coordinates of the center-of-mass, and second, using vector equations for the position of the center-of-mass.

1. **Scalar calculations:** Let \((x_{cm}, y_{cm})\) be the coordinates of the mass-center. Then from the definition of mass-center,

\[
x_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}
\]

\[
= \frac{1 \text{ kg} \cdot 1.25 \text{ m} + 2 \text{ kg} \cdot 2 \text{ m} + 3 \text{ kg} \cdot 0.75 \text{ m}}{1 \text{ kg} + 2 \text{ kg} + 3 \text{ kg}}
\]

\[
= \frac{7.5 \text{ kg} \cdot \text{ m}}{6 \text{ kg}} = 1.25 \text{ m}.
\]

Similarly,

\[
y_{cm} = \frac{\sum m_i y_i}{\sum m_i} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}
\]

\[
= \frac{1 \text{ kg} \cdot 3 \text{ m} + 2 \text{ kg} \cdot 2 \text{ m} + 3 \text{ kg} \cdot 0.5 \text{ m}}{1 \text{ kg} + 2 \text{ kg} + 3 \text{ kg}}
\]

\[
= \frac{8.5 \text{ kg} \cdot \text{ m}}{6 \text{ kg}} = 1.42 \text{ m}.
\]

Thus the center-of-mass is located at the coordinates \((1.25 \text{ m}, 1.42 \text{ m})\).

2. **Vector calculations:** Let \(\vec{r}_{cm}\) be the position vector of the mass-center. Then,

\[
m_{tot}\vec{r}_{cm} = \sum_{i=1}^{3} m_i \vec{r}_i = m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3
\]

\[\Rightarrow \vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_1 + m_2 + m_3}\]

Substituting the values of \(m_1, m_2, \) and \(m_3, \) and \(\vec{r}_1 = 1.25 \text{ m} \hat{i} + 3 \text{ m} \hat{j}, \ \vec{r}_2 = 2 \text{ m} \hat{i} + 2 \text{ m} \hat{j}, \) and \(\vec{r}_3 = 0.75 \text{ m} \hat{i} + 0.5 \text{ m} \hat{j},\) we get,

\[
\vec{r}_{cm} = \frac{1 \text{ kg} \cdot (1.25 \hat{i} + 3 \hat{j}) \text{ m} + 2 \text{ kg} \cdot (2 \hat{i} + 2 \hat{j}) \text{ m} + 3 \text{ kg} \cdot (0.75 \hat{i} + 0.5 \hat{j}) \text{ m}}{(1 + 2 + 3) \text{ kg}}
\]

\[
= \frac{(7.5 \hat{i} + 8.5 \hat{j}) \text{ kg} \cdot \text{ m}}{6 \text{ kg}}
\]

\[
= 1.25 \text{ m} \hat{i} + 1.42 \text{ m} \hat{j}
\]

which, of course, gives the same location of the mass-center as above.

\[\vec{r}_{cm} = 1.25 \text{ m} \hat{i} + 1.42 \text{ m} \hat{j}\]
SAMPLE 2.42 Center of mass of a bent bar: A uniform bar of mass 4 kg is bent in the shape of an asymmetric 'Z' as shown in the figure. Locate the center-of-mass of the bar.

Solution Since the bar is uniform along its length, we can divide it into three straight segments and use their individual mass-centers (located at the geometric centers of each segment) to locate the center-of-mass of the entire bar. The mass of each segment is proportional to its length. Therefore, if we let \( m_2 = m_3 = m \), then \( m_1 = 2m \); and \( m_1 + m_2 + m_3 = 4m = 4 \text{ kg} \) which gives \( m = 1 \text{ kg} \). Now, from Fig. 2.88,

\[
\overline{r}_1 = \ell \hat{i} + \ell \hat{j} \\
\overline{r}_2 = 2\ell \hat{i} + \frac{\ell}{2} \hat{j} \\
\overline{r}_3 = (2\ell + \frac{\ell}{2}) \hat{i} = \frac{5\ell}{2} \hat{i} \\
\]

So,

\[
\overline{r}_{\text{cen}} = \frac{m_1 \overline{r}_1 + m_2 \overline{r}_2 + m_3 \overline{r}_3}{m_{\text{tot}}} \\
= \frac{2m(\ell \hat{i} + \ell \hat{j}) + m(2\ell \hat{i} + \frac{\ell}{2} \hat{j}) + m(\frac{5\ell}{2} \hat{i})}{4m} \\
= \frac{\frac{\ell}{4}(13\ell + 5\ell)}{4\ell} \\
= \frac{0.5m}{8}(13\ell + 5\ell) \\
= 0.012m\hat{i} + 0.312m\hat{j} \\
\overline{r}_{\text{cen}} = 0.012m\hat{i} + 0.312m\hat{j}
\]

Geometrically, we could find the center-of-mass by considering two masses at a time, connecting them by a line and locating their mass-center on that line, and then repeating the process as shown in Fig. 2.89.

The center-of-mass of \( m_2 \) and \( m_3 \) (each of mass \( m \)) is at the mid-point of the line connecting the two masses. Now, we replace these two masses with a single mass 2m at their mass-center. Next, we connect this mass-center and \( m_3 \) with a line and find their combined mass-center at the mid-point of this line. The mass-center just found is the center-of-mass of the entire bar.
SAMPLE 2.43 Shift of mass-center due to cut-outs: A 2 m × 2 m uniform square plate has mass \( m = 4 \text{ kg} \). A circular section of radius 250 mm is cut out from the plate as shown in the figure. Find the center-of-mass of the plate.

**Solution** Let us use an \( xy \)-coordinate system with its origin at the geometric center of the plate and the \( x \)-axis passing through the center of the cut-out. Since the plate and the cut-out are symmetric about the \( x \)-axis, the new center-of-mass must lie somewhere on the \( x \)-axis. Thus, we only need to find \( x_{cm} \) (since \( y_{cm} = 0 \)). Let \( m_1 \) be the mass of the plate with the hole, and \( m_2 \) be the mass of the circular cut-out. Clearly, \( m_1 + m_2 = m = 4 \text{ kg} \). The center-of-mass of the circular cut-out is at \( A \), the center of the circle. The center-of-mass of the intact square plate (without the cut-out) must be at \( O \), the middle of the square. Then,

\[
m_1 x_{cm} + m_2 x_A = m x_O = 0
\]

\[
\Rightarrow x_{cm} = -\frac{m_2}{m_1} x_A.
\]

Now, since the plate is uniform, the masses \( m_1 \) and \( m_2 \) are proportional to the surface areas of the geometric objects they represent, i.e.,

\[
\frac{m_2}{m_1} = \frac{\pi r^2}{\ell^2 - \pi r^2} = \frac{\pi}{(\ell^2 - \pi r^2)}.
\]

Therefore,

\[
x_{cm} = -\frac{m_2}{m_1} d = -\frac{\pi}{(\ell^2 - \pi r^2)} d
\]

\[
= -\frac{\pi}{(\frac{250 \text{ mm}}{250 \text{ mm}})^2 - \pi} \cdot 0.5 \text{ m}
\]

\[
= -25.81 \times 10^{-3} \text{ m} = -25.81 \text{ mm}
\]

Thus the center-of-mass shifts to the left by about 26 mm because of the circular cut-out of the given size.

\[
x_{cm} = -25.81 \text{ mm}
\]

**Comments:** The advantage of finding the expression for \( x_{cm} \) in terms of \( r \) and \( \ell \) as in eqn. (2.42) is that you can easily find the center-of-mass of any size circular cut-out located at any distance \( d \) on the \( x \)-axis. This is useful in design where you like to select the size or location of the cut-out to have the center-of-mass at a particular location.
SAMPLE 2.44 Center of mass of two objects: A square block of side 0.1 m and mass 2 kg sits on the side of a triangular wedge of mass 6 kg as shown in the figure. Locate the center-of-mass of the combined system.

Solution The center-of-mass of the triangular wedge is located at $h/3$ above the base and $l/3$ to the right of the vertical side. Let $m_1$ be the mass of the wedge and $\overline{r}_1$ be the position vector of its mass-center. Then, referring to Fig. 2.93,

$$
\overline{r}_1 = \frac{l}{3} \hat{i} + \frac{h}{3} \hat{j}.
$$

The center-of-mass of the square block is located at its geometric center $C_2$. From geometry, we can see that the line $AE$ that passes through $C_2$ is horizontal since $\angle OAB = 45^\circ$ ($h = l = 0.3$ m) and $\angle DAE = 45^\circ$. Therefore, the coordinates of $C_2$ are $(d/\sqrt{2}, h)$. Let $m_2$ and $\overline{r}_2$ be the mass and the position vector of the mass-center of the block, respectively. Then,

$$
\overline{r}_2 = \frac{d}{\sqrt{2}} \hat{i} + h \hat{j}.
$$

Now, noting that $m_1 = 3m_2$ or $m_1 = 3m$, and $m_2 = m$ where $m = 2$ kg, we find the center-of-mass of the combined system:

$$
\overline{r}_{cm} = \frac{m_1 \overline{r}_1 + m_2 \overline{r}_2}{m_1 + m_2} = \frac{3m(\frac{l}{3} \hat{i} + \frac{h}{3} \hat{j}) + m(\frac{d}{\sqrt{2}} \hat{i} + h \hat{j})}{3m + m} = \frac{6h(\frac{l}{3} \hat{i} + \frac{h}{3} \hat{j})}{4m} = \frac{1}{4}(\frac{d}{\sqrt{2}} \hat{i} + h \hat{j}) = \frac{1}{4}(\frac{0.1}{\sqrt{2}} \hat{i} + 0.3 \hat{j}) = \frac{0.1 m}{4} \hat{i} + \frac{0.3 m}{4} \hat{j} = 0.093 m \hat{i} + 0.150 m \hat{j}.
$$

Thus, the center-of-mass of the wedge and the block together is slightly closer to the side OA and higher up from the bottom OB than $C_1 (0.1$ m, $0.1$ m). This is what we should expect from the placement of the square block.

Note that we could have, again, used a 1-D calculation by placing a point mass $3m$ at $C_1$ and $m$ at $C_2$, connected the two points by a straight line, and located the center-of-mass $C$ on that line such that $CC_2 = 3CC_1$. You can verify that the distance from $C_1 (0.1$ m, $0.1$ m) to $C (0.093$ m, $0.15$ m) is one third the distance from $C$ to $C_2 (0.071$ m, $0.3$ m).
Mass moment of Inertia problems:-

Dynamics Book Section 7.2, Solved examples 7.5 and 7.6