Conservation Equations

Fundamental concepts

Now that the fluid properties \((p, \rho, \vec{v}, \tau)\) have been reviewed, our next task is to consider the physical laws that underly fluid mechanics.

These are conservation equations of physics that tell us for a fixed system of particles, which quantities are constant in time, and how certain quantities change with time.

They are based on the following fundamental principles:

**Mass** is invariant in time

**Momentum** is changed by a force: \(\vec{F} = \frac{d}{dt}(m\vec{v})\)

**Energy** is conserved: it is neither created nor destroyed.

(This law is not important for Aero 301. It is, however, in Aero 303.)
You may ask: What about the \textbf{ideal gas law}? Is that one of our fundamental conservation equations?

\textbf{No}: It is a \textbf{State Equation}

\textbf{State equations} describe some sort of statistical outcome of the fundamental conservation equations on a scale we are not concerned with (i.e., the result of molecular dynamics).

\textit{The general approach}:

1. Determine a suitable model of the fluid.

2. Apply the \textbf{fundamental principles} to the model.

3. Obtain mathematical equations which describe the physics of the flow.
Conservation of Mass

Conservation equations are often written (usually by physicists) in a Lagrangian sense for a fixed system of particles.

Therefore, the mass conservation law (in words) is: “The mass of a fixed system of particles is invariant in time”.

Mathematical translation:

\[
\frac{d}{dt} \int \int \int_{V(t)} \rho(x,y,z,t) \, dV = 0
\]

where \( V(t) \) is continuously deformed to always contain the same set of particles.

However, we do not want to keep track of individual particles, so we cannot have a fixed system of particles. Instead, consider a fixed volume and keep track of how mass (particles) flows in and out.

So we change from a fixed set of particles to a fixed volume.
**Fixed volume approach:**

Consider a finite control volume fixed in space with the fluid moving through it.

Since mass is conserved, then:

\[
\text{Net mass flow out of } \mathcal{V} \text{ through } \mathcal{S} = \text{Time rate of change of mass inside } \mathcal{V}
\]

What is the net mass flow? Consider the mass flow through a \( d\mathcal{S} \) on the surface of the volume, \( \mathcal{V} \). We have

\[
\rho \mathbf{v} \cdot \hat{n} \, d\mathcal{S}
\]

where \( \hat{n} \) is an **outward pointing, surface-normal unit vector**.

When summed over all elemental surfaces \( d\mathcal{S} \) we have
\[ \int \int \int_S \rho \vec{v} \cdot \hat{n} \, dS \]

By convention, positive mass flow is \textit{out} and negative is \textit{into} the control volume \( V \).

What is the time rate of change of mass inside \( V \)? Total mass in \( V \) is given by

\[ \int \int \int_V \rho \, dV \]

Rate of \textit{decrease} of mass inside \( V \) is given by

\[ -\frac{\partial}{\partial t} \int \int \int_V \rho \, dV \]

Note that \( \rho \) varies with position and time (i.e., \( \rho(x,y,z,t) \)). It is not an average across the entire volume.
Since the volume is fixed with time, the partial derivative can move inside the integral

\[ -\int \int \int \rho \frac{\partial}{\partial t} d\mathcal{V} \]

Equating mass flow out with time rate of change we arrive at the integral form of the mass-conservation equation written for a fixed control volume as

\[ \int \int \int \rho \frac{\partial}{\partial t} d\mathcal{V} + \int \int \rho \mathbf{v} \cdot \hat{n} d\mathcal{S} = 0 \]
Conservation of Momentum

Newton’s Second Law:

\[ \text{Force} = \text{Time rate of change of momentum} \]

Mathematically, this means:

\[ \vec{F} = \frac{d}{dt}(m\vec{v}) \]

Once again, this applies to a set collection of particles — not a convenient arrangement. Considering again of our fixed finite volume \( \mathcal{V} \) and thinking about momentum flux into and out of the volume, then we have in words:

\[ \vec{F} = \text{Net mom. out of } \mathcal{V} \text{ through } \mathcal{S} + \text{Time rate of change of mom. inside } \mathcal{V} \]
We write the equations in terms of **momentum per volume**: $\rho \vec{v}$

The net flow of momentum **out of** $\mathcal{V}$ through $\mathcal{S}$ is

$$\int \int \mathcal{S} \rho \vec{v} (\vec{v} \cdot \hat{n}) d\mathcal{S}$$

The momentum inside $\mathcal{V}$ at any instant is given by

$$\int \int \int \mathcal{V} \rho \vec{v} d\mathcal{V}$$

The time rate of change of momentum due to unsteady fluctuations of flow properties inside of $\mathcal{V}$ is therefore

$$\frac{\partial}{\partial t} \int \int \int \mathcal{V} \rho \vec{v} d\mathcal{V}$$
Combining these expressions we can therefore write

\[ \int \int \int_V \rho \vec{v} \partial (\vec{v}) \partial t dV + \int \int_{S} \rho \vec{v} (\vec{v} \cdot \hat{n}) dS = \vec{F} \]

Note that this is a vector equation!

*Types of forces \( \vec{F} \)*

What are the forces on the right hand side of the equation?

\[ \vec{F} = \vec{F}_{\text{body}} + \vec{F}_{\text{surface}} \]

**Body forces,** \( \vec{F}_{\text{body}} \) affect every particle in the volume.

Examples: gravity, electromagnetic forces; any force which act at a distance from our volume \( \Omega \).

**Surface forces,** \( \vec{F}_{\text{surf}} \) are the forces that result from the contact of the fluid on the boundary with the environment (solid surface or other fluid).

These are pressure and shear forces on the control surface \( \mathcal{S} \).
In practical aerodynamics, gravity is the body force we typically encounter. How do we deal with it?

We integrate the **gravity force per volume** over the volume

$$\vec{F}_{\text{body}} = \int \int \int_V \rho \vec{g} \, dV$$

There are two components of the surface force, both are **stresses, or, forces per area**, so we must integrate over the surface area to get a force

**Pressure** always acts in the $-\hat{n}$ direction

**Viscous Shear Stress** cannot specify direction *a priori*

The fact that we cannot specify force directions *a priori* makes surface forces a little harder to deal with because the **direction** of the surface force vector depends on the orientation of the surface and the velocities at the surface.
There are two ways we can write the surface force:

\[ \vec{F}_{\text{surf}} = \begin{cases} 
- \iint_{\mathcal{S}} p \hat{n} \, d\mathcal{S} & \text{if viscosity is unimportant} \\
\vec{F}_{\text{surf}} & \text{if viscosity is important}
\end{cases} \]

If viscosity is important we do nothing. In the context of this class, \( \iint \vec{F}_{\text{surf}} \) (the drag or something similar) becomes what we would solve for.

When do we worry about viscosity? Viscous stress terms look like

\[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]

so viscosity is important if \( \mu \) times a velocity gradient is large.

The concept of large is defined by a dimensional analysis.

We can usually choose boundaries for our control volume that minimize or eliminate velocity gradients.

Combining expressions for the net momentum out of the control volume \( \mathcal{V} \) and time rate
of change of momentum inside $\mathcal{V}$ with expressions for the forces, then we can write the whole momentum equation as

\[
\int \int \int_{\mathcal{V}} \frac{\partial (\rho \vec{v})}{\partial t} d\mathcal{V} + \int \int_{\partial \mathcal{V}} \rho \vec{v} (\vec{v} \cdot \hat{n}) d\mathcal{S} = \\
- \int \int_{\mathcal{S}} p \hat{n} d\mathcal{S} + \int \int \int_{\mathcal{V}} \rho \vec{g} d\mathcal{V} + \vec{F}_{\text{visc}}.
\]

This is the momentum equation in integral form. Note that it is a vector equation. Just like the continuity equation, it has the advantage of relating aerodynamic phenomena over a finite region of space $\mathcal{V}$, without being concerned about the precise details of what is going on at a point within the flow.
Example: Integral Lift and Drag on an Airfoil in a Wind Tunnel

Problem Assumptions:

- Steady and incompressible
- Gravity is negligible
- No viscous effects on boundaries 1–4
- 2D flow with depth $b$ into page
- $v = 0$ on boundaries 1 & 3 (the boundaries are far from the model)
Example: Part II of the cylinder drag problem

Problem Assumptions:

• Steady & Incompressible
• Gravity is negligible
• Pressure is $p_\infty$ on boundaries 1–4
• No viscous effects on boundaries 1–4
• 2D flow with depth $b$ into page
Transforming from integral to differential formulations

Over the past couple lectures we have been working with an integral (control volume) formulation of the conservation equations.

This is a great approach for getting general force results (e.g., overall lift and drag) when we would rather not worry about the details of the mechanism that leads to the forces we measure.

However, in this class we also care about the mechanisms. Why does a particular wing produce the lift and drag it does? How would a different shape change the results?

If an integral approach hides the details because the equations apply to a large area, a differential approach exposes the details because differential equations hold at every point in a flowfield.

Q: How do we transform our integral governing equations to differential governing equations?

A: Calculus III
In particular, we make use of two theorems:

<table>
<thead>
<tr>
<th>Divergence Theorem</th>
<th>Gradient Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \iiint \vec{q} \cdot \hat{n} dS = \iint \nabla \cdot \vec{q} dV ]</td>
<td>[ \iiint \vec{q} \hat{n} dS = \iint \nabla q dV ]</td>
</tr>
</tbody>
</table>

In both theorems, \( \hat{n} \) is an outward-pointing surface-normal vector.

This is a scalar equation in which \( \vec{q} \) is a vector.

This is a vector equation in which \( q \) is a scalar.

What is the **physical** significance of these theorems?
Converting the Mass Equation

Recall the integral form of the mass (continuity) equation

\[
\iint \int_V \frac{\partial \rho}{\partial t} \, dV + \iint_S \rho (\vec{v} \cdot \hat{n}) \, dS = 0
\]

We recognize the \( \rho \vec{v} \) in the second term to be the \( \vec{q} \) in the divergence theorem.

Using the divergence theorem, we can replace the \( \iint \) term with an \( \iiint \) term:
...and combine these into a single triple integral

\[
\iiint_{V} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right] dV = 0
\]

We immediately recognize that this integral must equal zero for any \( V \), regardless of how small it is. Therefore, the integrand must be equal to zero and we obtain the differential vector equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]

This is the most general form of the mass-conservation (continuity) equation in differential form.

If we put \( \rho = \text{const} \) into the continuity equation we obtain
\[ \nabla \cdot \vec{v} = 0 \]

which is an elegant result for incompressible flows.

The momentum equation is a little trickier because (1) we need to use both the gradient theorem and the divergence theorem and (2) it is a vector equation, not a scalar equation.

We start by writing the momentum equation as

\[
\int \int \int_V \frac{\partial (\rho \vec{v})}{\partial t} dV + \int \int_{\partial V} \rho \vec{v} (\vec{v} \cdot \hat{n}) d\mathcal{S} = \int \int \int_V \rho \vec{g} dV - \int \int_{\partial V} p \hat{n} d\mathcal{S} + \vec{F}_{\text{visc}}.
\]

The aim is to write all the terms as volume integrals. It is convenient to write this vector equation in terms of three scalar equations. Recognizing that \( \vec{v} = (u, v, w) \) we can expand
each term into three vector components

\[
\text{unsteady} = \left( \begin{array}{c}
\int \int \int_V \rho u \frac{\partial \rho u}{\partial t} \, dV \\
\int \int \int_V \rho v \frac{\partial \rho v}{\partial t} \, dV \\
\int \int \int_V \rho w \frac{\partial \rho w}{\partial t} \, dV
\end{array} \right)
\]

\[
\text{convection} = \left( \begin{array}{c}
\int \int \int_S \rho u (\vec{v} \cdot \hat{n}) \, dS \\
\int \int \int_S \rho v (\vec{v} \cdot \hat{n}) \, dS \\
\int \int \int_S \rho w (\vec{v} \cdot \hat{n}) \, dS
\end{array} \right)
\]

Now, recognizing that \( \rho u \vec{v} \) is \( \vec{q} \), we can then apply the divergence theorem to the convective terms to give
convection = \left\{ \begin{array}{l}
\int\int\int_V \nabla \cdot (\rho \mathbf{u} \mathbf{v}) \, dV
\end{array} \right\}

The pressure term uses the gradient theorem

pressure = \int\int\int_V \nabla \cdot (\rho \mathbf{u} \mathbf{v}) \, dV = \left\{ \begin{array}{l}
\int\int\int_V \frac{\partial p}{\partial x} \, dV
\end{array} \right\}

The gravity term is easy...
Let us consider the $x$ component (and neglect $\vec{F}_{\text{visc.}}$). Assemble all of the $x$ components into one equation involving all triple integrals over $\mathcal{V}$

$$
\int \int \int_{\mathcal{V}} \frac{\partial (\rho u)}{\partial t} d\mathcal{V} + \int \int \int_{\mathcal{V}} \nabla \cdot (\rho u \vec{v}) d\mathcal{V} = -\int \int \int_{\mathcal{V}} \frac{\partial p}{\partial x} d\mathcal{V} + \int \int \int_{\mathcal{V}} \rho g_x d\mathcal{V}
$$

And using the same argument as before about the generality of $\mathcal{V}$

$$
\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \vec{v}) = -\frac{\partial p}{\partial x} + \rho g_x
$$

This is called the **conservative form** of the differential representation of the conservation of momentum (in the $x$ direction).

Now expand the first two terms and group terms with $\rho$ and $u$

$$
\rho \left( \frac{\partial u}{\partial t} + \vec{v} \cdot \nabla u \right) + u \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right] = -\frac{\partial p}{\partial x} + \rho g_x
$$
You will notice that the $u[...]$ term is identically zero according to the differential form of the continuity equation.

Therefore, the inviscid, compressible momentum equations are

$$
\rho \left( \frac{\partial u}{\partial t} + (\vec{v} \cdot \nabla) u \right) = -\frac{\partial p}{\partial x} + \rho g_x
$$

$$
\rho \left( \frac{\partial v}{\partial t} + (\vec{v} \cdot \nabla) v \right) = -\frac{\partial p}{\partial y} + \rho g_y
$$

$$
\rho \left( \frac{\partial w}{\partial t} + (\vec{v} \cdot \nabla) w \right) = -\frac{\partial p}{\partial z} + \rho g_z
$$

This set of equations are called the Euler Equations. If they included viscous terms they would be called the Naiver–Stokes Equations (in which case we would have needed to include $\vec{F}_{\text{visc.}}$).
The Material Derivative

When we transform from an integral to differential formulation we see a common form emerge:

$$\frac{\partial \text{something}}{\partial t} + (\vec{v} \cdot \nabla) \text{something}$$

Because this shows up in all of the conservation equations' differential forms we give it the shorthand

$$\frac{\partial \text{something}}{\partial t} + (\vec{v} \cdot \nabla) \text{something} = \frac{D\text{something}}{Dt}$$

and call this complicated derivative the **material derivative** or sometimes the **material derivative**.

However, the material derivative has a very important **physical** meaning: The material derivative is a concept that links the time rate of change of an element's properties (a Lagrangian concept) to derivatives of the field properties (an Eulerian concept).
Imagine an element whose density is known because it is at a known position, $\vec{x}(t)$, in a known density field, $\rho(x,y,z,t)$.

What is $D\rho/Dt$ for the element as it moves through the field?

Therefore, we have

$$\frac{D\rho}{Dt} = \frac{d\rho}{dt} \bigg|_{\text{element}} = \frac{\partial \rho}{\partial t}\bigg|_{\text{local}} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z}$$

$D\rho/Dt$ is the time rate of change of density of a given fluid element as it passes through space.

This idea works for other properties as well: $\vec{a} = D\vec{v}/Dt$, etc.
Example: Consider the acceleration of a fluid element \( \vec{a} = \left( \frac{Du}{Dt}, \frac{Dv}{Dt} \right) \) in a 2D stagnation-point flow:

\[
\begin{align*}
  u &= k x \\
  v &= -k y
\end{align*}
\]

1. Verify that the flow satisfies continuity
2. What is the location of the stagnation point (\( \vec{v} = 0 \))?
3. Find the acceleration \( \vec{a} \)
4. What is the pressure gradient \( \partial p / \partial x \) along the wall?
Streamlines

In the preceding example, the figure shows streamlines, which are *lines whose tangent at any point is in the direction of the velocity vector* or, from a different perspective, lines that fluid elements do not cross.

Because fluid elements move along streamlines, *streamlines are parallel to \( \vec{v} \) at every point in the field.*

What equations describe these lines?

In 2D
Consider the velocity on the streamline \( \mathbf{V}(x, y, z, t) \), and by definition \( \mathbf{V}(x, y, z, t) \) is parallel \( \mathbf{ds} \). Hence the following is valid

\[
\mathbf{ds} \times \mathbf{V} = 0
\]

where

\[
\mathbf{ds} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}
\]

\[
\mathbf{V} = u \mathbf{i} + v \mathbf{j} + w \mathbf{k}
\]

making the cross product we get

\[
\begin{align*}
wdy - vdz &= 0 \\
u dz - wdx &= 0 \\
wdx - udy &= 0
\end{align*}
\]

which are three differential equations for the streamline. If we knew \( u, v, w \) as functions of \( x, y, z \) then we could integrate them and obtain the equation for the streamline \( f(x, y, z) = 0 \).
To give physical meaning, consider the flow in 2D

\[
\frac{dy}{dx} = \frac{v}{u} \quad \text{or} \quad u dy - v dx = 0
\]

Thus, the equations merely state mathematically that the velocity vector is always tangent to the streamline.
What are the equations for the streamlines in stagnation-point flow?

If we know $\vec{v}(\vec{r})$ we can apply the boxed equations on the previous page and integrate to find the streamlines. Different constants of integration give the different streamlines in the flow.

Note that one of these streamlines runs along the solid surface.

In general, **body surfaces are streamlines** because the no-penetration boundary condition is equivalent to the fluid-does-not-cross-a-streamline condition.

Q: If we release a tagged fluid element at a point on the streamline does it follow the streamline?

A: Yes if...
Bernoulli’s Equation

So far, we have already developed the fundamental integral and differential equations which relate fluid velocities to pressures. We usually solve for velocities first, then pressures, then integrate pressures to find net forces (e.g., lift & drag) and moments.

Often, the step of using the velocity field to find $p$ can be simplified by recognizing that we are most interested in $p$ along body streamlines. So, let us integrate the momentum equation to find $p$ along a particular streamline.

The equation is $\rho \frac{D\vec{v}}{Dt} = -\nabla p + \rho \vec{g}$ for inviscid flow (Euler’s equation since we have neglected viscosity).

Restricting ourselves to steady flow, $D\vec{v}/Dt$ becomes...
Extract just the $x$ component and multiply each term by $dx$...

$$u \frac{\partial u}{\partial x} dx + v \frac{\partial u}{\partial y} dx + w \frac{\partial u}{\partial z} dx = -\frac{1}{\rho} \frac{\partial p}{\partial x} dx + g_x dx$$

Now recognize our **streamline relationships**: $v dx = u dy$ and $w dx = u dz$...
or

$$u \left( \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \right) = -\frac{1}{\rho} \frac{\partial p}{\partial x} dx + g_x dx$$

From calculus, you know that the differential of $u$ can be written as

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

Hence we can write

or

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$
Thus, for all three components of the momentum equation we have

\[
\frac{1}{2}d(u^2) = -\frac{1}{\rho} \frac{\partial p}{\partial x} dx
\]

\[
\frac{1}{2}d(v^2) = -\frac{1}{\rho} \frac{\partial p}{\partial y} dy + g_y dy
\]

\[
\frac{1}{2}d(w^2) = -\frac{1}{\rho} \frac{\partial p}{\partial z} dz + g_z dz
\]

Adding all these up we have \( \frac{1}{2}d(V^2) = -\frac{1}{\rho} dp + \vec{g} \cdot \vec{d}\vec{x} \)

(Do you see how we got the \( dp \)?)
Neglecting body forces, we have

\[ dp = -\rho V dV \]

This equation takes on a special form when we consider an \textit{incompressible} flow, where \( \rho = \text{constant} \). Integrating between two points on a streamline gives

Integrating gives the celebrated Bernoulli's equation

\[ p_1 + \frac{1}{2} \rho V_1^2 = p_1 + \frac{1}{2} \rho V^2 \]

or
This is a very famous and useful equation in fluid dynamics. The physical significance is that:

*When the velocity increases, then pressure decreases, and when the velocity decreases, then the pressure decreases.*

Note that the dimensions of Bernoulli’s equation is energy per unit volume. Hence, it is simply an expression for mechanical energy in an incompressible flow, stating that the work done by pressure forces is equal to the change in kinetic energy of the flow.