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STORE-INDUCED LIMIT CYCLE OSCILLATIONS AND INTERNAL RESONANCES IN AEREOELASTIC SYSTEMS

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ABSTRACT

Several advanced fighter aircraft designs that carry underwing stores have encountered limit cycle oscillations at speeds well below the predicted flutter velocity. Flight tests suggest that the onset of these limit cycle oscillations is related to the type and location of the stores carried. However, no mechanism has been forwarded to conclusively explain the onset of these oscillations. Internal resonance is a dynamical phenomenon due to the nonlinear coupling between modes of motion. The possibility of nonlinear behavior such as internal resonance in a system can be overlooked in analysis, as linearization of the equations of motion often eliminates crucial nonlinear terms. Internal resonance has been shown to be a theoretically possible occurrence in aerelastic systems. Nonlinear analysis and simulation of a two degree-of-freedom aerelastic system pursuant to experimental investigation is described herein. It is suggested that even kinematic nonlinearities inherent in the system should be retained for accurate nonlinear results. High levels of damping complicate the analysis of internal resonance in the experimental system. Nonlinear analysis may facilitate the examination of internal resonance or other nonlinear phenomena in a basic aerelastic system.

INTRODUCTION

Several fighter aircraft designs have encountered limit cycle oscillations (LCOs) for certain external store configurations. Such LCOs may cause excessive fatigue of an airframe, shortening the service life of the aircraft. Additionally, such oscillations generate unacceptable workloads for pilots, leading to pilot fatigue and increasing the difficulty of performing critical tasks. (See Bunton and Denegri[1]).

This phenomenon is not confined to historical or obsolete aircraft; rather, it is prevalent in more advanced aircraft as well. Denegri[2] and Yurkovitch[3] describe flight test results in which store-related LCOs are encountered for the F-16 and the F/A-18, respectively.

Denegri[2] notes that for the F-16, linear flutter analysis may properly identify the frequency of the oscillation, but fails to predict the oscillation amplitude or the oscillation onset velocity. Similarly, Yurkovitch[3] notes for the F/A-18 that even when taking into account factors such as the movement of fuel in the wings, linear analysis estimates the onset velocity of the LCOs to be higher than the velocities in question.

Since the velocity estimation provided for the so-called "wing-store flutter" is not conservative, and since these LCOs are undesirable, linear aerelastic analysis has proven to be insufficient to define a safe and useful flight envelope. For the F-16 and the F/A-18, this has

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meant that a usable flight envelope can only be defined by extensive (and thus costly) flutter flight testing. Denegri[9] notes that a nonlinear flutter analysis method is needed in order to successfully predict the onset of LCOs.

Aeroelasticity, by definition, is the study of the interaction between aerodynamic, structural, and inertial forces. Aeroelasticity is inherently nonlinear, due to the nonlinearities in aerodynamics (including unsteady sources and stall), the structure (including system free play, hardening-type response, and damping mechanisms), and inertia (due to the location and distribution of masses). Limit cycles are also an inherently nonlinear phenomenon. Thus, it is logical to conclude that accurate modeling and prediction of a nonlinear phenomenon such as a limit cycle in a nonlinear aeroelastic system should be approached via nonlinear analysis.

Another nonlinear phenomenon which may be present in the aeroelastic system is the phenomenon known as internal resonance (IR). Internal resonance is caused by nonlinear coupling between modes of motion. Linear analysis tends to eliminate these nonlinear coupling terms, which often have little effect on the system. However, when the frequencies of the modes are nearly commensurate (ω₁ ≈ 2ω₂, 2ω₅ ≈ ω₂, ω₁ ≈ 3ω₂, etc.), the modes may interact strongly, with the possible transfer of energy from one mode to the other. (See Nayfeh and Mook[9]). Figure 1 demonstrates the response of a system with a quadratic nonlinear coupling. If the system is not “tuned” with a 2:1 frequency ratio, the response is simply oscillatory. When the system is tuned, a periodic amplitude modulation will occur.

Gilliatt[9] and Chang, et al.[10] independently demonstrated, via different approaches, that internal resonance is theoretically possible in an aeroelastic system. The frequencies of oscillation in an aeroelastic system are dependent on the dynamic pressure. As the dynamic pressures increase, the wing bending frequency and the wing torsion frequency may become tuned such that the frequencies are commensurate and the presence of an internal resonance is made obvious. Gilliatt limited analysis to considering the nonlinearity associated with wing stall, whereas Chang, et al., focused on a time-dependent drag force. Figure 2 illustrates the simulated effect of a cubic stall-associated aerodynamic nonlinearity on wing response. The linearized response is also plotted to illustrate that linear theory fails to predict the growth in the response near the 3:1 frequency ratio.

The investigation of internal resonance by Gilliatt was motivated by the experimental findings of Cole[11]. Cole, while conducting wind tunnel experiments, encountered LCOs at speeds lower than those predicted. It is possible that the system tested by Cole was exhibiting an internal resonance response, as opposed to classical flutter, which can be predicted by linear theory.

Stearman, et al.[12] have looked at internal resonance with external stores. However, this work focused on using statistical techniques to analyze flight test data from an aircraft with various store configurations.

EXPERIMENT BACKGROUND

The objective of this investigation was to examine the possible link between store-induced LCOs and internal resonance in a simplified aeroelastic system. One goal of this investigation is to conduct benchmark experiments to show a wing-with-store in an internal resonance condition. For this reason, the analytical model used is based on the Nonlinear Aeroelastic Test Apparatus (NATA) at Texas A&M University. The NATA is described below.

The NATA is an experimental system developed for the study of linear and nonlinear aeroelastic response, either open-loop or with active control of the response. The apparatus allows the attached wing section to move in two modes—translation and rotation, in order to model an aeroelastic system. Instrumentation allows the measurement of pitch (rotation) and plunge (translation). The data acquisition system also provides for the actuation of a control surface for studies in flutter and limit cycle oscillation suppression. The NATA is integrated into the 2’x3’ low-speed wind tunnel at Texas A&M University. Figure 3 is a diagram of the NATA.

The aeroelastic system is modeled as a wing limited to motion in two degrees of freedom, pitch and plunge. The bending mode of the wing, represented by the plunge motion, is governed by a translational spring. The torsional mode of the wing, represented by the pitching motion, is governed by a rotational spring. No assumptions are immediately made about the linearity of the springs—they may have linear or nonlinear responses. The wing section, as it is depicted in Figure 4, is assumed to be infinitely rigid, i.e. all elastic response is due to the illustrated translational and rotational springs.
This two degree of freedom model is sufficient for the study of classical bending-torsion flutter. When the elastic response is nonlinear, this model can be used to study the limit cycle oscillations associated with flutter.

The NATA was developed as a means of conducting experiments based on the two degree of freedom aeroelastic model. (See O’Neil et al. and Strganac et al. The rotational stiffness is governed by springs attached to a cam attached to the shaft. A circular cam will provide a linear elastic response. A non-circular cam (e.g., a parabolic shape) will provide a prescribed nonlinear response. The shaft, cam, and springs are attached to a carriage. This carriage translates on linear bearings, allowing the wing section to plunge. The motion of this carriage is also governed by springs attached to either a linear or nonlinear cam. Constraints on the apparatus limit the amplitude of both the pitch motion and the plunge motion in order to prevent damage to the wind tunnel or the apparatus.

Typically, the NATA is equipped with a circular cam to govern the plunge stiffness and a parabolic cam to provide a spring hardening response in the pitch mode. There is evidence that some aircraft show such a spring-hardening nonlinearity in the torsion mode.

**ANALYSIS**

We have derived equations of motion for the test apparatus that include a model store attached to the wing. The equations may be expressed in the following form:

\[
\begin{align*}
\dot{y} + & \left[ (m_r - a) m_r + (t_r - a) m_t + (t_c - a) m_c \right] \cos(\alpha) \\
\dot{\alpha} + & \left[ (m_r - a) m_r + (t_r - a) m_t + (t_c - a) m_c \right] \sin(\alpha) \\
\dot{\dot{y}} + & \left[ (m_r - a) m_r + (t_r - a) m_t + (t_c - a) m_c \right] \cos(\alpha) \\
\dot{\dot{\alpha}} + & \left[ (m_r - a) m_r + (t_r - a) m_t + (t_c - a) m_c \right] \sin(\alpha)
\end{align*}
\]

where \( y \) and \( \alpha \) represent the plunge and pitch coordinates of the attached wing section, respectively. Overdots represent time derivatives. \( k_r \) and \( k_c \)

represent stiffness functions, which may be linear or nonlinear. \( c_r \) and \( c_c \) represent the viscous damping coefficients, though it should be noted that Coulomb-type damping is also present in experimentation. The parameter \( m_r \), represents the total mass of the system, which is the sum of the masses of the wing section, (\( m_w \)), pitch cam, (\( m_t \)), model store, (\( m_c \)), and the plunge carriage. Since the plunge carriage translates with the wing section, its mass must be taken into account. Similarly, \( t_r \), represents the sum of the mass moments of inertia of the wing section (\( I_w \)), the pitch cam (\( I_t \)), and the model store (\( I_c \)), each about their respective centers of gravity. \( b \) represents the semichord of the wing, and \( x_y \) represents the nondimensional distance between the elastic axis and the center of mass, as shown in Figure 4. \( L \) and \( M \) represent the lift and moment due to aerodynamic forces, with respect to the elastic axis. The parameters \( x_r \), \( y_r \), \( z_r \), \( x_s \), \( y_s \), \( z_s \), \( x_t \), \( y_t \), \( z_t \), \( x_c \), and \( y_c \), are nondimensional distances relating the location of the centers of mass of the wing, store, and pitch cam to the midpoint of the mid chord, as shown in Figure 5. \( \alpha \) and \( \theta \) represent the nondimensional distance between the midpoint of the mid chord and the elastic axis.

For this study, the aerodynamics are assumed to be quasi-steady, with the lift and moment given as follows:

\[
L = \rho U^2 b S C_{L_{\alpha}} \alpha_{\text{eff}}
\]

\[
M = \rho U^2 b S C_{M_{\alpha}} \alpha_{\text{eff}} (\frac{1}{2} + a)
\]

where \( \alpha_{\text{eff}} \) is given by

\[
\alpha_{\text{eff}} = \alpha + \frac{\dot{y}}{U} + \frac{b (\frac{1}{2} - a) \dot{\alpha}}{U}
\]

This model is appropriate for the low velocities and reduced frequencies examined. With the lift and moment equations substituted into the equations (1) and (2), the resulting equations are nondimensionalized to facilitate analysis. The following definitions are made in order to conveniently express the nondimensional equations:

\[
m_r = m_w + m_t + m_c + m_{\text{car}}
\]

\[
I_t = I_w + I_s + I_c
\]
\[ \mu = \frac{m_i}{rb^2 \pi S} \quad (8) \]

\[ v_r = \frac{m_r}{m_i} \quad v_z = \frac{m_z}{m_i} \quad v_c = \frac{m_c}{m_i} \quad (9a,b,c) \]

\[ r_a = \sqrt{\frac{I_i}{m_i b^2}} \quad (10) \]

\[ c_1 = \frac{c_a b}{m_i U} \quad c_2 = \frac{c_a a^2 b}{I_i U} \quad (11a,b) \]

\[ K_y^2 = \frac{b}{U m_i} k_y \quad K_\alpha^2(\alpha) = \frac{b}{I_i} \frac{k_\alpha(\alpha)}{U} \quad (12a,b) \]

\[ h = \frac{y}{b} \quad (13) \]

\[ \tau = \frac{Ut}{b} \quad (14) \]

Note that \( h \) represents the nondimensional plunge coordinate, and \( \tau \) represents nondimensional time. Also, it should be noted that the definitions of some of the above parameters are similar to those typically found in aeroelastic analysis. The value of \( \mu \) is based on \( m_i \), rather than the mass of the wing only, and \( r_a^2 \) is based on the quantity \( I_i \), rather than the moment of inertia about the elastic axis. Caution should be taken when comparing this system to systems having only a wing mass or systems having a single lumped mass.

For this analysis, \( k_y \) is a constant, but \( k_\alpha \) is a nonlinear function of \( \alpha \), as shown in equation (15). The nonlinearity present in the NATA experiment is predominantly a cubic nonlinearity, but it is modeled as a fifth order polynomial so that simulations of the system more closely match experimental results. Figure 6 shows the measured stiffness of the nonlinear pitch cam versus the rotational displacement, and the polynomial fit used to model this response.

\[ k_\alpha(\alpha) = k_1 + k_2 \alpha + k_3 \alpha^2 + k_4 \alpha^3 + k_5 \alpha^4 \quad (15) \]

Consequently, the parameter \( K_\alpha \) is given as follows:

\[ K_\alpha(\alpha) = K_\alpha \sqrt{k_1 + k_2 \alpha + k_3 \alpha^2 + k_4 \alpha^3 + k_5 \alpha^4} \quad (16) \]

With the above substitutions, the nondimensionalized equations of motion for the wing-store-cam system may be written as

\[ h'' + ((r_x - a) v + (s_x - a) v_x + (t_x - a) v_c) \cos(\alpha) a'' \]

\[ -((r_y - a_y) v + (s_y - a_y) v_x + (t_y - a_y) v_c) \sin(\alpha) a'' \]

\[ -((r_z - a_z) v + (s_z - a_z) v_x + (t_z - a_z) v_c) \sin(\alpha) a'' \]

\[ -((r_x - a_x) v + (s_x - a_x) v_x + (t_x - a_x) v_c) \cos(\alpha) a'' \]

\[ \left( c_1 + \frac{c_a b}{\mu \pi} \right) a'' + \left( \frac{3}{2} - a - \frac{c_{1a}}{\mu \pi} \right) A' + \frac{c_{1a} b}{\mu \pi} = 0 \quad (17) \]

The primes in the above equation denote differentiation with respect to nondimensional time, \( \tau \).

These equations are inconveniently long, so some additional substitutions are defined.

\[ \gamma_0 = \frac{C_{1a}}{\mu \pi} \]

\[ \gamma_1 = (r_x - a) v + (s_x - a) v_x + (t_x - a) v_c \]

\[ \gamma_2 = (r_y - a_y) v + (s_y - a_y) v_x + (t_y - a_y) v_c \quad (18) \]

\[ \gamma_3 = (r_y - a_y) v + (s_y - a_y) v_x + (t_y - a_y) v_c \]

\[ \gamma_4 = c_1 \]

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\[ \gamma_5 = c_2 \]

where \( \gamma_0 \) is related to the aerodynamics of the system; \( \gamma_1, \gamma_2, \) and \( \gamma_3 \) are related to the geometry of the system; and \( \gamma_4 \) and \( \gamma_1 \) are related to the structural viscous damping present in the system.

For some purposes it is desirable to eliminate the \( \sin(\alpha) \) and \( \cos(\alpha) \) terms from equations (17) and (18). However, such nonlinearities should not be casually discarded when other nonlinearities are fully retained. (This is discussed in greater detail below). Therefore, it would be inappropriate to simply apply the small angle assumption. Instead, the sine and cosine terms are replaced with Taylor series expansions accurate to third order, as the structural nonlinearity is assumed to be primarily a cubic nonlinearity in \( \alpha \).

\[
\cos(\alpha) = 1 - \frac{1}{2} \alpha^2 \tag{20}
\]

\[
\sin(\alpha) = \alpha - \frac{1}{6} \alpha^3 \tag{21}
\]

Now, one substitutes equations (19)-(21) into equations (17) and (18) to obtain the simplified form of the equations of motion. These equations are expressed as follows, and it is noted that they are highly nonlinear:

\[
\begin{bmatrix}
1 & r_x(\alpha + \frac{1}{6} \alpha^3) & r_y(\alpha + \frac{1}{6} \alpha^3) & \frac{1}{\alpha} \frac{d}{d\alpha} \left[ \frac{1}{\alpha} \right] \\
\frac{r_x}{r_y} & r_x(r_x - \frac{1}{6} \alpha^3) & r_y(r_y - \frac{1}{6} \alpha^3) & \frac{1}{\alpha} \frac{d}{d\alpha} \left[ \frac{1}{\alpha} \right] \\
\frac{r_y}{r_x} & r_y(r_y - \frac{1}{6} \alpha^3) & r_x(r_x - \frac{1}{6} \alpha^3) & \frac{1}{\alpha} \frac{d}{d\alpha} \left[ \frac{1}{\alpha} \right] \\
K_2^2 & K_2^2 \gamma_4 & K_2^2 \gamma_4 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\alpha} \\
\dot{\beta} \\
\dot{\gamma} \\
\dot{\delta}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} \tag{22}
\]

**EXAMINATION OF NONLINEAR SYSTEM RESPONSE**

The general response of this system has been studied experimentally. (See O’Neil et al.\(^{10}\) and Srganic et al.\(^{11}\)). In the current investigation, simulations have also been used to study the system response. Figures 7 and 8 illustrate the simulated response of a wing-store-cam system based on the NATA structure. It should be noted that these simulations were based on the dimensional equations of motion, and included sine and cosine terms rather than the Taylor-series approximations of equations (20) and (21).

Figure 7 shows the response of the system to a plunge initial condition when at a flow velocity below the linear flutter speed. Figure 8 shows the response to the same system given the same initial conditions but with a slightly higher flow velocity, which is slightly above the linear flutter speed. In Figure 7, the initial plunge disturbance induces motion in the pitch mode, but both motions eventually damp out. By contrast, in Figure 8, the initial plunge disturbance again induces motion in the pitch mode, but the motion does not damp out. Transient behavior disappears within a short time and both modes become entrained in a limit cycle oscillation. Figures 9(a) and 9(b) show phase portraits for the plunge and pitch modes in the limit oscillation, respectively. Recall that the limit cycle, which is clearly illustrated by the phase portraits, indicates the nonlinear nature of the system. Without the nonlinearity introduced by the torsional stiffness function, one would most likely expect the motion to grow without bound, as in linear flutter.

However, even if the torsional stiffness of the system were linear, the kinematics of the problem would still introduce nonlinearities into the equations of motion. In order to simplify analysis, it might be tempting to linearize the problem enough to eliminate these nonlinearities. However, further consideration of the problem, along with a sample simulation of the wing-store-cam system based on the NATA structure, indicates that such simplifications may be inadvisable.

Consider \( \gamma_1 \), defined in equation (19c), which relates the "vertical" distance of the wing, store, and cam centers of mass relative to the chord line. For a system with only a wing, this number would be quite small. (See Figure 5). The value would be zero for the symmetric airfoil used in the NATA experiment. However, a large store might have a center of gravity located well below the chord line. Thus the value of \( \gamma_1 \) alone may not be negligible in this case.

However, one might argue that \( \gamma_1 \) and \( \gamma_1 \), which both appear in the kinematic nonlinearities of equation (22), are still combined with other terms that make the overall effect of the kinematic nonlinearities negligible. Indeed, in many cases simulations show that the overall response of a wing-store-cam system is very similar whether or not the kinematic nonlinearities are included. However, Figures 10 and 11 show the respective plunge responses of systems with and without kinematic nonlinearities. The systems were given identical initial conditions. Note that in Figure 10, there is a small, transient growth in amplitude immediately following the initial decay in motion.
Figure 11 shows no evidence of such a transient increase in energy.

We avoid here the discussion of whether or not an internal resonance was present in this system to focus on the differences between the inclusion and exclusion of the kinematic nonlinearities. Of note is the fact that the system without the kinematic nonlinearities ignores some higher order nonlinear features of the system. In many cases, this may be an acceptable approximation. However, in cases where higher-order nonlinear features become evident, such approximation may lead to inaccuracies that cause the same features of the behavior to be overlooked.

In the course of this investigation, the response for the fully nonlinear wing-store-cam system over a wide range of conditions was examined for evidence of internal resonance. For the system without structural damping in the absence of aerodynamics, the amplitude modulation often associated with internal resonance was demonstrated. However, the presence of structural and aerodynamic damping complicates the motion of the system, making the identification of internal resonance more difficult. With a system based on parameters of the NATA, the amplitude of motion damps out very quickly, making it very difficult to examine slow changes in modal amplitude over time.

It would be beneficial to have a way in which to characterize exactly how important the kinematic nonlinearities to the system response. It would also be beneficial to have a methodology for identifying the presence of internal resonance in a system with relatively high levels of damping. Therefore, a nonlinear analysis of the system was undertaken.

**NONLINEAR ANALYSIS**

Thus far, the nonlinear analysis used on this system has been primarily based on the Method of Variation of Parameters. Gilliatt analyzed an aeroelastic system using the Method of Multiple Scales (see Nayfeh and Mook and Bush). However, Gilliatt was analyzing a specialized set of equations in which the kinematic nonlinearities had been eliminated via colocation of the elastic axis and center of mass of the wing. (This is equivalent to choosing $\gamma_1=\gamma_2=\gamma_3=0$ in equation (22)). This investigation is specifically seeking to examine the kinematic nonlinearities, for, as discussed above, these may be significantly altered by the presence of a store on the wing. Thus, the kinematic nonlinearities, and the additional coupling terms they produce in the mass and damping matrices, must be retained.

The system in equation (22) has coupling terms in the mass and damping matrices. Additionally, even when completely linearized, the system is coupled by aerodynamics. These complications have made it difficult to apply the Method of Multiple Scales to this system. In this investigation, a practical, workable way of applying the Method of Multiple Scales to the system is of interest.

The Krylov-Bogoliubov method of averaging was considered, but this method is valid only to first order (see Nayfeh and Mook and Bush), and would therefore be insufficient for analyzing this system. However, the Krylov-Bogoliubov method, like most other methods of averaging, begins with the Method of Variation of Parameters. While the Method of Variation of Parameters alone will not result in analytical solutions for $h$ and $\alpha$, it does appear to yield insight into the behavior of the system.

The Method of Variation of Parameters, as outlined in Nayfeh and Mook and Bush, begins with assuming that $h$ and $\alpha$ are of the following form:

$$ h(\tau) = a_1(\tau) \cos(\omega_1 \tau + \beta_1(\tau)) $$

$$ \alpha(\tau) = a_2(\tau) \cos(\omega_2 \tau + \beta_2(\tau)) $$

where $a_1(\tau)$ and $a_2(\tau)$ are time-varying amplitudes of the modes, $\omega_1$ and $\omega_2$ are linearized natural frequencies, and $\beta_1(\tau)$ and $\beta_2(\tau)$ are time-varying nonlinear corrections to the frequencies. These equations are also written as

$$ h(\tau) = a_1(\tau) \cos(\phi_1) $$

$$ \alpha(\tau) = a_2(\tau) \cos(\phi_2) $$

where

$$ \phi_1 = \omega_1 \tau + \beta_1(\tau) $$

$$ \phi_2 = \omega_2 \tau + \beta_2(\tau) $$

and
\[ \omega_n = K_v = \frac{b^2}{U^2} \frac{k_v}{m_i} \]  
\[ \omega_a = \frac{K_{a_\phi} \sqrt{\frac{k_i}{\gamma_2 + r_a^2}}} \]  
(30)  
(31)

The original two functions of nondimensional time, \( h' \) and \( \alpha' \), have been transformed into four functions of nondimensional time, \( a_1, a_2, \beta_1, \) and \( \beta_2 \). Since the number of time functions has been increased, any convenient constraint can be applied to the equations, without approximation or loss of generality. A standard constraint to impose is to constrain the velocity to be of the same form as the original expression, written as follows:

\[ h'(\tau) = -\omega_a a_1(\tau) \sin(\omega_a \tau + \beta_1(\tau)) \]  
\[ \alpha'(\tau) = -\omega_a a_2(\tau) \sin(\omega_a \tau + \beta_2(\tau)) \]  
(32)  
(33)

Equations (16), (23) and (24), and (32) and (33) can be substituted into the system of equation (22) and expanded. By manipulating the two original equations, one can derive equations in terms of \( a_1(\tau), \beta_1(\tau), a_2(\tau), \) and \( \beta_2(\tau) \), and the derivatives of these four functions with respect to nondimensional time. At this point, a method of averaging would typically be used to derive approximate expressions for \( a_1(\tau), \beta_1(\tau), a_2(\tau), \) and \( \beta_2(\tau) \), as both Nayfeh and Mook[40] and Bush[11] explain. However, these equations were examined in an unaltered form. While these equations are highly nonlinear, these equations are linear with respect to the time derivatives of the functions. Using linear algebra, one can arrive at a set of equations of the following form:

\[
\begin{bmatrix}
  a_1' \\
  \beta_1' \\
  a_2' \\
  \beta_2'
\end{bmatrix} = \begin{bmatrix}
  f_1(a_1, \beta_1, a_2, \beta_2) \\
  f_2(a_1, \beta_1, a_2, \beta_2) \\
  f_3(a_1, \beta_1, a_2, \beta_2) \\
  f_4(a_1, \beta_1, a_2, \beta_2)
\end{bmatrix}
\]  
(34)

This should not imply that these equations are easily manipulated or that they result in simple expressions. Rather, without any form of approximation, these equations are far too lengthy to reproduce here.

However, it is instructive to look at the general form of the equations. The general form of the \( a_i'(\tau) \) equation can be written has terms of the type

\[ f(a_1, \beta_1, a_2, \beta_2) e^{\pm \varphi} \]

and so on, up to about the fifth order. Note that while the \( \phi_1 \pm \phi_2 \) term does appear, no \( \phi_2 \pm 3 \phi_1 \) term appears.

If the \( \phi_1 \) and \( \phi_2 \) terms were just linear functions of nondimensional time (i.e., \( \phi_1 = \omega_1 \tau \)), it could be shown that the nonlinear coupling would occur when \( \phi_1 = \phi_2, \phi_1 = 2 \phi_2, \phi_1 = 3 \phi_2, \) etc., because of the above exponential terms. (See Bajaj et al. [12]) It is not suggested that the relationship developed herein can be treated that simply. However, the question of whether any interesting phenomena occur when \( \phi_1 = \phi_2, \phi_1 = 2 \phi_2, \phi_1 = 3 \phi_2, \phi_1 = 3 \phi_2, \) etc., is raised. This question has not yet been answered strictly via analysis, but attempts to answer the question using numerical approaches have been pursued.

**RESULTS**

A program was developed in the MATLAB\textsuperscript{6} environment to numerically integrate equation (34) over nondimensional time. System constants were based on the NATA parameters. Structural damping was eliminated and velocity was minimized to reduce aerodynamic damping. (A nonzero velocity was necessary to avoid numerical singularities in the code). The nonlinear torsional stiffness relationship was unaltered, but the linear plunge stiffness was altered to provide different frequency responses. Each simulation was given the same pitch and plunge initial conditions. Figures 12 - 14 show normalized amplitudes \( a_1(\tau) \) and \( a_2(\tau) \) over nondimensional time \( \tau \). The amplitudes are normalized so that they can be compared on the same plot. It is important to note that the amplitudes, as shown, were numerically smoothed via averaging to remove high frequency features in order to elucidate the lower frequency effects. Note that as the amplitudes were calculated, so were \( \beta_1(\tau) \) and \( \beta_2(\tau) \), the nonlinear frequency corrections. Thus the values \( \phi_1 \) and \( \phi_2 \) could be calculated for each step in nondimensional time.
Three cases were computed. The first case, as illustrated in Figure 12, the plunge stiffness was chosen such that $\omega_r = 3\omega_n$. When the nonlinear frequency relationship is observed, the ratio of $\phi_1:\phi_2$ is initially near 1.6, but increases to roughly 1.7 by the end of the indicated time period. For the second case, shown in Figure 13, a plunge stiffness was chosen such that the ratio of $\phi_1:\phi_2$ was approximately 3:1 for the entire time period. This corresponded to a linear natural frequency relationship of $\omega_r = 5.6\omega_n$. A third case, shown in Figure 14, was chosen so that neither the linear natural frequency ratio nor the nonlinear frequency relationship was near an integer ratio. The stiffness was chosen such that $\omega_r = 4.3\omega_n$, or about halfway between the first and second cases. The ratio of $\phi_1:\phi_2$ for this case was initially near 2.4, increasing to about 2.6 over the indicated time.

As one might observe from Figure 12, the case where $\omega_r = 3\omega_n$ seems to show small shifts in amplitude between the modes of motion, indicating an exchange of energy. By comparison, Figure 14, the case where $\omega_r = 5.6\omega_n$, shows little, if any, of such behavior, indicating that there is, at best, a weak mechanism for transferring energy between modes. However, in Figure 13, the case in which $\phi_1 = 3\phi_2$, there is clearly a slow modulation in amplitude, suggesting an exchange in energy between the modes. Some damping may be present in the system, slowly decreasing the total energy in the system. Note that internal resonance is typically defined in terms of commensurate linear natural frequencies (see Nayfeh and Mook[40]), as is the case in Figure 12. Therefore, the question of whether the case shown in Figure 13 is actually indicative of an internal resonance, or some other, perhaps related, nonlinear dynamic phenomenon is raised.

It was noted that in the simulations of the system, when the system has a high level of damping, amplitude modulation is difficult to detect. The same is true for the direct study of the time-varying amplitudes. The amplitudes still damp out quickly, making it difficult to examine any slow variations in amplitude. Solutions to this difficulty are still under investigation.

CONCLUSIONS

Aerodynamic systems are inherently nonlinear due to nonlinearities in aerodynamics, structures, and inertia. Limit cycle oscillations, including store-induced limit cycle oscillations present in certain advanced aircraft, are a nonlinear phenomenon. Internal resonance is another nonlinear phenomenon that is possible in aerelastic systems.

While it is still uncertain whether or not there is a link between store-induced limit cycle oscillations and internal resonance, it is clear that the analysis and solution of these phenomena must be studied using nonlinear analysis. Elimination or over-simplification of even kinematic nonlinearities may lead to significant differences in predicted nonlinear response in an aerelastic system.

The amplitude modulation often associated with the presence of an internal resonance is difficult to study in systems with high levels of damping. Amplitudes tend to damp out too quickly to show slowly varying modulations in amplitude. Nonlinear analysis may provide a means of analyzing internal resonance and perhaps other related nonlinear phenomena in a system with high damping.

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REFERENCES


FIGURES

Figure 1. Time response of a nonlinear system. (One mode is plotted above the axis, the other below). For a linear frequency ratio of 1.5:1 (left),

the system exhibits an oscillatory response. For the same system, "tuned" to a 2:1 frequency ratio, the pure oscillatory response is replaced by a periodic amplitude modulation.

Figure 2. Evidence of an internal resonance is observed as a wing passes through the 3:1 frequency ratio. Linear theory fails to exhibit this response. (From Gilliatt[5]).

Figure 3. Schematic view of the Nonlinear Aeroelastic Test Apparatus (NATA).

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Figure 4. Theoretical representation of the aeroelastic system, highlighting key parameters.

Figure 5. Illustration of simplified wing-store aeroelastic system. (The horizontal placement of the nonlinear pitch cam CG is exaggerated for clarity).

Figure 6. Measured and modeled moment versus alpha for the nonlinear pitch cam.

Figure 7. Sample simulation of wing-store-cam system, below linear flutter speed.
Figure 8. Sample simulation of wing-store-cam system. The system is above the linear flutter speed and is entrained in a limit cycle oscillation.

Figure 9. Plunge and pitch phase portrait plots for the system of Figure 8.

Figure 10. Sample simulation of wing-store-cam system, with all nonlinearities included.

Figure 11. Sample simulation of wing-store-cam system, with kinematic nonlinearities eliminated.
Figure 12. Normalized modal amplitudes for a system with $\omega_\gamma = 3\omega_\alpha$.

Figure 13. Normalized modal amplitudes for a system with $\omega_\gamma = 5.6\omega_\alpha$, $\phi = 3\phi_\gamma$.

Figure 14. Normalized modal amplitudes for a system with $\omega_\gamma = 4.3\omega_\alpha$. 

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