Two-Way Orbits

Ossama Abdelkhalik* and Daniele Mortari†
Department of Aerospace Engineering, Texas A&M University, College Station, TX 77843, USA

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Abstract. This paper introduces a new set of compatible orbits, called the “Two-way orbits”, whose ground track path is a closed-loop trajectory that intersect itself, in some points, with tangent intersections. The spacecraft passes over these tangent intersections once in a prograde and once in a retrograde mode. The general mathematical model to design the Two-way orbits is presented for the specific case where the tangent points are experienced at the orbit extremes: perigee and apogee. As for the general case, the Two-way orbits conditions are formulated and numerically solved. Results show that, in general, Two-way orbits could be formed at a general tangency point. However, if the two spacecraft are in a Flower Constellation, then the only possibility for the tangency point is at the perigee of one spacecraft and the apogee of the other. Using these Two-way orbits, this paper also introduces the Two-way Constellations, that have one spacecraft prograde and one retrograde, passing simultaneously over the tangent intersections.

Keywords: Orbits, Two Way Orbits, Flower Constellations

1. Introduction

The theory of Flower Constellations (FCs, see Mortari, Wilkins, and Bruccoleri) has introduced the concept of a spacecraft constellation built using orbits that are compatible with respect to an assigned rotating reference frame. This implies that all the spacecraft, in this rotating frame, follow the same continuous closed-loop trajectory. In particular, when the reference frame is chosen to be Earth-Centered Earth-Fixed (ECEF) reference frame, then the FC spacecraft all follow the same relative trajectory (space track) in ECEF and, consequently, the same continuous closed-loop ground track. In general, the ground track is made of prograde and retrograde parts, depending if the local ground

* Graduate Research Assistant, Department of Aerospace Engineering, Texas A&M University, 620C H.R. Bright Building, 3141 TAMU, College Station, Texas 77843-3141. Tel. (979) 458-0550, email: omar@tamu.edu
† Associate Professor, Department of Aerospace Engineering, 701 H.R. Bright Building, Room 741A, Texas A&M University, 3141 TAMU, College Station, Texas 77843-3141. Tel. (979) 845-0734, Fax (979) 845-6051. email: mortari@tamu.edu

For information on the Flower Constellations see also (Bruccoleri, Wilkins, and Mortari), (Park, Wilkins, Bruccoleri and Mortari), (Park, Ruggieri and Mortari), (Wilkins), (Wilkins, Bruccoleri, and Mortari), and (Wilkins and Mortari.)

track longitude increases or decreases with the time, respectively. Also, the ground track (for a compatible orbit) is a continuous closed-loop line that intersect itself in several points. These intersections can be characterized by the angle between the ground velocity of the two intersecting parts. When this angle is equal to $\pi$, then the intersecting point is a tangent intersecting point and the two intersecting parts are, locally, one prograde and another retrograde, respectively, over the intersecting point on the Earth surface. This describes the concept of the Two-way orbits.

In the Two-way orbits the relative trajectory will have (at least) one tangent intersecting point. This implies that it is possible to build a special Flower Constellations with a spacecraft moving along the tangent prograde direction and another spacecraft moving along the retrograde tangent direction. In particular, it is possible to phase the spacecraft in such a way they will pass over the tangent point simultaneously.

Two cases will be considered here. The first is the special Two-way orbits where the tangency point is at the perigee of one spacecraft and the apogee of the other. The second case is the general Two-way orbits where the tangency point is any general point on the trajectories of the two spacecraft.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Two-way orbit example.}
\end{figure}

In this paper, conditions on the orbits parameters such that they constitute a Two-way orbit are derived. The second section will briefly review the theory of Flower Constellations. The third section will review the orbit compatibility conditions. In the fourth section, the case of special two-way orbits is considered. Condition on the orbits inclination is derived to have Two-way orbits. A plot is generated relating the inclination vs. the eccentricity for specified values of the semimajor axis. The fifth section will develop similar analysis for the general Two-way
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orbits; However, the solution for the formulation is done numerically for this case. Algorithm for the numerical solution is presented in the sixth section. The last section considers the compatibility of the developed conditions with the Flower Constellation theory. Results show that the derived conditions are compatible with the FCs only in the special Two-Way Orbits.

2. Flower Constellations

We shall briefly present some background material regarding the subject of the (FCs). The FCs are characterized by the following properties:

1. Axial symmetric dynamics,
2. Satellite orbits are general compatible orbits,
3. Constellation axis can be re-oriented,
4. Multiple constellations (constellations$^2$),
5. Repeated-repeated ground track (repeated$^2$),
6. Relative Sun-Synchronous orbits,
7. Morphing constellations,
8. Secondary paths,
9. FC Visualization and Analysis Tool (FCVAT)$^2$.
   a) Software written in JAVA and JAVA-3D,
   b) Works on every machine and OS with 3D capabilities,
   c) Easy input/output,
   d) (ingoing) STK input,
   e) Any planet, Sun, and fictitious planet.

The spacecraft orbits are General compatible orbits: each satellite follows the same relative trajectory with respect to a rotating reference frame. When ECEF is selected, then the orbits are compatible with respect to the Earth rotating frame (repeated ground track).

Admissible positions / phasing\(^3\): the closed-loop relative trajectory intersects the associated (general compatible) inertial orbit in many points. A subset of these points identifies the admissible positions for a spacecraft to belong to the same relative trajectory. The logic used to distribute the satellites in the admissible positions encompasses all the possible different distribution (patent pending on admissible positions and phasing logic).

3. Orbit Compatibility

Consider an Earth-Centered Earth-Fixed (ECEF) system of coordinates identified by \( \mathcal{E} = \{ O, \hat{e}_x, \hat{e}_y, \hat{e}_z \} \), where the origin \( O \) is at the center of the Earth, \( \hat{e}_x \) on the equatorial plane at Greenwich meridian, \( \hat{e}_z \) aligned with Earth’s spin axis, and \( \hat{e}_y = \hat{e}_z \times \hat{e}_x \) to form a right-handed reference frame.

One orbit is called compatible with respect to the Earth\((\text{Carter, 1991})\) when the spacecraft trajectory in \( \mathcal{E} \) constitutes a closed-loop relative trajectory. A compatible orbit, which is sometime inappropriately called repeated ground track orbits\(^4\), is defined as the orbit whose orbital nodal period \( T_{\Omega} \) (node to node) satisfies the relationship

\[
N_p T_{\Omega} = N_d T_{\Omega G}
\]

where \( N_p \) and \( N_d \) are two integer numbers indicating the number of orbit periods and the number of the Earth rotational periods to repeat, and where \( T_{\Omega G} \) is the Greenwich nodal period, which has been defined by Carter\((\text{Carter, 1991})\) as

\[
T_{\Omega G} = \frac{2\pi}{\dot{\alpha}_{\oplus} - \dot{\Omega}}
\]

where \( \dot{\alpha}_{\oplus} = 7.29211585530 \times 10^{-5} \) rad/sec is the rotation rate of the Earth and \( \dot{\Omega} \) is the nodal regression of a satellite’s orbit plane caused by perturbations such as the Earth’s oblateness. In particular, \( T_r = N_p T_{\Omega} \) is the period of repetition on the relative trajectory.

Equations (1) and (2) allow us to write

\[
T_{\Omega} = \left( \frac{2\pi}{\dot{\alpha}_{\oplus} - \dot{\Omega}} \right) \frac{N_d}{N_p} = \left( \frac{2\pi}{\dot{\alpha}_{\oplus} - \dot{\Omega}} \right) \xi
\]

\(^3\) Algorithm under review by D. Mortari and M.P. Wilkins for intellectual property with the Technology Licensing Office, TAMU 3369, 707 Texas Ave, College Station, TX 77843-3369.

\(^4\) Any two equatorial orbits have the same repeated ground track but, in general, they do not follow the same relative trajectory in ECEF, that is, they are not compatible.
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where $\xi = N_d/N_p$ is the rational compatibility parameter. Equation (3) tells us that, for every distinct value of $\xi$, there is a different nodal period $T_\Omega$, associated with Earth’s compatible orbits. However, this equation can also be seen from a different perspective: for a given value of $\xi$, an arbitrary orbit (with nodal period $T_\Omega$ and nodal rate $\dot{\Omega}$) can be seen as compatible with a fictitious Earth that rotates with angular velocity

$$\dot{\alpha} = \dot{\Omega} + \frac{2\pi}{T_\Omega} \xi$$

Therefore, every orbit can be seen compatible with an associated Earth-Centered Rotating (ECR) system of coordinates that rotates at the angular velocity provided by Eq. (3). The final result is that any Earth-compatible orbit is compatible with infinite ECR reference frames.\(^5\) The compatibility concept is a relative concept, which is referring always to a rotating reference frame. Thus, if we consider a different rotating reference frame, then there will be a definition of orbit compatibility with respect to this reference frame.

4. The Special “Two-way” Orbits

The condition for two satellites to have tangent ground tracks at a point is to have parallel Earth-relative velocities at that point. The Earth-relative velocity, $\vec{v}$, is the velocity of the satellite with respect to an Earth rotating system of coordinates.

$$\vec{v} = \vec{V} - \vec{V}_E$$

where $\vec{V}$ is the satellite velocity in ECI and $\vec{V}_E$ is the local geographical velocity evaluated at radius $\vec{r}$ in ECI. The transformation matrix between inertial and orbital reference frames is ($C \equiv \cos$ and $S \equiv \sin$)

$$R^T = \begin{bmatrix} C_\Omega C_\omega - C_i S_\Omega S_\omega & -C_\Omega S_\omega - C_i S_\Omega C_\omega & S_i S_\Omega \\ S_\Omega C_\omega + C_i C_\Omega S_\omega & -S_\Omega S_\omega + C_i C_\Omega C_\omega & -S_i C_\Omega \\ S_\omega S_i & C_\omega S_i & C_i \end{bmatrix}$$

This matrix allows to evaluate inertial ($r_i$) from orbital ($r_o$) directions, and viceversa

$$r_i = R^T r_o \quad \iff \quad r_o = R r_i$$

Based on this result, it is possible to build Flower Constellations (Mortari, Wilkins, and Bruccoleri, 2004) highlighting the existence and the dynamics of multiple secondary paths (Wilkins, 2004).
In particular, position and velocity are transformed accordingly with

\[ r_i = \frac{p}{1 + e \cos \varphi} \begin{bmatrix} \cos \Omega \cos(\omega + \varphi) - \sin \Omega \sin(\omega + \varphi) \cos i \\ \sin \Omega \cos(\omega + \varphi) + \cos \Omega \sin(\omega + \varphi) \cos i \\ \sin(\omega + \varphi) \sin i \end{bmatrix} \]  

(8)

while the velocity in orbital reference frame is expressed as

\[ v_o = \sqrt{\frac{\mu}{p}} \begin{bmatrix} -\sin \varphi \\ e + \cos \varphi \\ 0 \end{bmatrix} \]  

(9)

Let us consider, for simplicity, two eccentric orbits having the apsidal lines lying on the equatorial plane (\( \omega = 0 \)). Under this condition, the velocity of the first at perigee is

\[ \vec{V}_{p1} = (e + 1) \sqrt{\frac{\mu}{p}} \begin{bmatrix} -\sin \Omega_1 \cos i \\ \cos \Omega_1 \cos i \\ \sin i \end{bmatrix} \]  

(10)

while at apogee of the second orbit the velocity is

\[ \vec{V}_{a2} = (e - 1) \sqrt{\frac{\mu}{p}} \begin{bmatrix} -\sin \Omega_2 \cos i \\ \cos \Omega_2 \cos i \\ \sin i \end{bmatrix} \]  

(11)

The local geographical velocity is

\[ \vec{V}_E = \vec{\omega}_E \times \vec{r} \]  

(12)

where \( \vec{\omega}_E \) is the Earth angular velocity. Specializing Eq. (12) for the perigee position and orbit #1 we obtain

\[ \vec{V}_{Ep1} = \begin{bmatrix} 0 \\ 0 \\ \omega_E \end{bmatrix} \times \frac{p}{1 + e} \begin{bmatrix} \cos \Omega_1 \\ \sin \Omega_1 \\ 0 \end{bmatrix} = \frac{p \omega_E}{e + 1} \begin{bmatrix} -\sin \Omega_1 \\ \cos \Omega_1 \\ 0 \end{bmatrix} \]  

(13)

while at apogee of orbit #2 the Earth velocity is

\[ \vec{V}_{Ea2} = \begin{bmatrix} 0 \\ 0 \\ \omega_E \end{bmatrix} \times \frac{p}{1 - e} \begin{bmatrix} -\cos \Omega_2 \\ -\sin \Omega_2 \\ 0 \end{bmatrix} = \frac{p \omega_E}{e - 1} \begin{bmatrix} -\sin \Omega_2 \\ \cos \Omega_2 \\ 0 \end{bmatrix} \]  

(14)

Substituting Eqs. (10) and (13) into Eq. (5) we obtain

\[ \vec{V}_p^R = (e + 1) \sqrt{\frac{\mu}{p}} \begin{bmatrix} -\sin \Omega_1 \cos i \\ \cos \Omega_1 \cos i \\ \sin i \end{bmatrix} - \frac{p \omega_E}{e + 1} \begin{bmatrix} -\sin \Omega_1 \\ \cos \Omega_1 \\ 0 \end{bmatrix} \]  

(15)
while Eqs. (11) and (14) into Eq. (5) we obtain

\[
\vec{V}^R_a = (e - 1) \sqrt{\frac{p}{p}} \begin{bmatrix}
- \sin \Omega_2 \cos i \\
\cos \Omega_2 \cos i \\
\sin i
\end{bmatrix} - \frac{p \omega E}{e - 1} \begin{bmatrix}
- \sin \Omega_2 \\
\cos \Omega_2 \\
0
\end{bmatrix}
\] (16)

In general, in order to have tangent ground tracks at the intersection, the two velocity vectors and the vector pointing at the intersection, \( \vec{R}_{eq} \), must be linearly dependent (they identify the plane passing through the origin of the coordinates and containing the two velocities \( \vec{V}^R_a \) and \( \vec{V}^R_p \)). In our case, since we are looking for tangency at equator and experienced at perigee/apogee, then our tangency condition can be substituted with the condition for the two velocity vectors, \( \vec{V}^R_a \) and \( \vec{V}^R_p \), of being parallel. This implies that we can write

\[
\vec{V}^R_a = k \vec{V}^R_p
\] (17)

where \( k \) is the proportionality constant whose value can be directly obtained from the third scalar identity of Eq. (17)

\[
k = \frac{e - 1}{e + 1}
\] (18)

This condition yields to the relationship

\[
\frac{-C_i S_{\Omega_1} V_p + \omega_E r_p S_{\Omega_1}}{C_i S_{\Omega_2} V_a - \omega_E r_a S_{\Omega_2}} = \frac{C_i C_{\Omega_1} V_p - \omega_E r_p C_{\Omega_1}}{-C_i C_{\Omega_2} V_a + \omega_E r_a C_{\Omega_2}}
\] (19)

and

\[
\frac{-C_i S_{\Omega_1} V_p + \omega_E r_p S_{\Omega_1}}{C_i S_{\Omega_2} V_a - \omega_E r_a S_{\Omega_2}} = \frac{V_p}{-V_a}
\] (20)

With little manipulation, Eq. (19) is satisfied iff

\[
\sin (\Omega_1 - \Omega_2) = 0
\] (21)

or

\[
V_a V_p C_i^2 - \omega_E C_i (r_p V_a + r_a V_p) + \omega_E^2 r_p r_a = 0
\] (22)

The latter case is refused because it does not satisfy the condition in Eq. (20) The result in Eq. (21) states that either \( \Omega_1 = \Omega_2 \), which is a trivial case where the two orbits are identical, or

\[
\Omega_1 - \Omega_2 = \pi
\] (23)

The latter gives the condition on the right ascension of ascending node for the two orbits. The condition in Eq. (20), after manipulation, implies that

\[
C_i = \frac{\omega_E \left( \frac{r_p}{V_p} S_{\Omega_1} - \frac{r_a}{V_a} S_{\Omega_2} \right)}{S_{\Omega_1} - S_{\Omega_2}}
\] (24)
From Eq. (23), we have \(\sin(\Omega_2) = -\sin(\Omega_1)\) then Eq. (24) simplifies to

\[
\cos i = \frac{\omega E}{2} \left[ \frac{r_p}{V_p} + \frac{r_a}{V_a} \right] - \frac{2e\omega E}{\sqrt{a(1-e^2)}} \sqrt{\frac{a^3}{\mu}}
\]  

(25)

where \(V_p = \sqrt{\frac{2\mu}{r_p} - \frac{\mu}{a}}\) and \(V_a = \sqrt{\frac{2\mu}{r_a} - \frac{\mu}{a}}\).

Equations (23) and (25) constitutes the necessary and sufficient conditions to have Two-way orbits.

Figure 2 shows Two-Way Orbits inclinations for different values of eccentricities and semi-major axis.

**Figure 2.** Two-Way Orbits inclinations for different values of eccentricities and semi-major axis

5. The General “Two-way” Orbits

We considered the case where the two orbits have similar shape, size and inclination. Moreover, we assumed zero argument of perigee and the intersection point occur at the perigee and apogee points of the two orbits. In this section we look at the same problem but the intersection point is any general point, not necessarily an apogee or perigee.

First we find the condition of having an intersection between the two ground tracks for two different orbits. An intersection between the ground tracks occurs if the two position vectors of the two satellites are
Substituting for the vectors \( \vec{r}_1 \) and \( \vec{r}_2 \) from Eq. (8) then we can get the following two conditions for an intersection to occur:

\[
\frac{\cos \Omega_1 \cos(\omega + \varphi_1) - \sin \Omega_1 \sin(\omega + \varphi_1) \cos i}{\cos \Omega_2 \cos(\omega + \varphi_2) - \sin \Omega_2 \sin(\omega + \varphi_2) \cos i} = \frac{\sin(\omega + \varphi_1)}{\sin(\omega + \varphi_2)} \tag{27}\]

and

\[
\frac{\sin \Omega_1 \cos(\omega + \varphi_1) + \cos \Omega_1 \sin(\omega + \varphi_1) \cos i}{\sin \Omega_2 \cos(\omega + \varphi_2) + \cos \Omega_2 \sin(\omega + \varphi_2) \cos i} = \frac{\sin(\omega + \varphi_1)}{\sin(\omega + \varphi_2)} \tag{28}\]

These two conditions can be simplified to the following form:

\[
\begin{bmatrix}
    C_i a_1 & -a_3 \\
    a_3 & C_i a_1
\end{bmatrix}
\begin{bmatrix}
    S_{\Omega_1} \\
    C_{\Omega_1}
\end{bmatrix}
= \begin{bmatrix}
    C_i a_1 & -a_4 \\
    a_4 & C_i a_1
\end{bmatrix}
\begin{bmatrix}
    S_{\Omega_2} \\
    C_{\Omega_2}
\end{bmatrix} \tag{29}
\]

where, \( a_1 = \sin(\omega + \varphi_1) \sin(\omega + \varphi_2) \), \( a_3 = \cos(\omega + \varphi_1) \sin(\omega + \varphi_2) \), and \( a_4 = \sin(\omega + \varphi_1) \cos(\omega + \varphi_2) \).

5.1. Observations

1. We notice that for the special case where \( a_3 = a_4 \), then either \( \Omega_1 = \Omega_2 \), which is obvious case, or \( C_i^2 a_1^2 + a_3^2 = 0 \) and this is satisfied only if \( \omega + \varphi_2 = n\pi \) where, \( n = 0, 1, \ldots \) This later case is the special case solved in the previous section.

2. If we eliminate \( C_i \) from Eq. (29) then we get

\[
\frac{a_4 S_{\Omega_2} - a_3 S_{\Omega_1}}{\Delta C} = \frac{a_3 C_{\Omega_1} - a_4 C_{\Omega_2}}{\Delta S} \tag{30}
\]

where \( \Delta C = C_{\Omega_1} - C_{\Omega_2} \) and \( \Delta S = S_{\Omega_1} - S_{\Omega_2} \). Then by rearrangement of Eq. (30) we can write

\[
\frac{a_4}{a_3} = \frac{\tan(\omega + \varphi_1)}{\tan(\omega + \varphi_2)} = \frac{C_{\Omega_1} \Delta C + S_{\Omega_1} \Delta S}{C_{\Omega_2} \Delta C + S_{\Omega_2} \Delta S} \tag{31}
\]

This can be further simplified to the form

\[
(a_3 + a_4) \{ \cos (\Omega_1 - \Omega_2) - 1 \} = 0 \tag{32}
\]

Which means that either \( \Omega_1 = \Omega_2 \), which is a trivial solution, or \( a_3 = a_4 \). The latter can be written in the form:

\[
\omega_1 + \varphi_1 + \omega_2 + \varphi_2 = n\pi \quad n = 0, 1, \ldots \tag{33}
\]
3. For given $\Omega_1$ and $\Omega_2$, Eq. (31) gives one relation between $\varphi_1$ and $\varphi_2$ at the intersection point. Another relation is required to find the two anomalies. The other relation should relate them through the time equation. The above relation only states that the intersection is possible between the two tracks but not necessarily implies that the both satellites will pass by this point at the same time. A second condition should relate the two anomalies through the time such that the two satellite will pass by the intersection point at the same time.

4. For the special case where $\Omega_1 - \Omega_2 = \pi$, the above equation reduces to $a_2/a_3 = -1$. This means that $\varphi_2 = \pi - \varphi_1$, which is the special case solved in the previous section.

The condition in Eq. (33) implies that the two trajectories of two satellites intersects at certain point; however, it does not imply that the two satellites will pass by this point both at the same time. To guarantee that both will pass by the intersection point at the same time, we introduce the following condition.

Assume the two satellites of interest intersect at time $t = t_i$, and writing the time equation for both satellites at the intersection point:

\[
(t_i - t_{p1}) n = \psi_{1i} - e \sin(\psi_{1i})
\]

(34)

\[
(t_i - t_{p2}) n = \psi_{2i} - e \sin(\psi_{2i})
\]

(35)

where $n$ is the mean motion. Then,

\[
\psi_{2i} - e \sin(\psi_{2i}) = \psi_{1i} - e \sin(\psi_{1i}) - n(t_{p2} - t_{p1})
\]

(36)

In order to find $t_{p1}$ and $t_{p2}$, we need to define the phasing between the two satellites. Recalling that for a Flower Constellation we have two phasing conditions (Mortari, Wilkins, and Bruccoleri, 2004). The first, for a two body case:

\[
M_{k+1}(0) = M_k(0) + 2\pi \frac{F_n}{F_d} \frac{n}{\omega_E}
\]

(37)

where $F_n$ and $F_d$ are integers defining the phasing of the two satellites. Now if we set, without loss of generality, $t_{p1} = -M_1(0)/n$ and $t_{p2} = -M_2(0)/n$, then the condition in Eq. (36) becomes:

\[
\psi_{2i} - e \sin(\psi_{2i}) = \psi_{1i} - e \sin(\psi_{1i}) + \left(2\pi \frac{F_n}{F_d} \frac{n}{\omega_E}\right)
\]

(38)

Eq. (33) and Eq. (38) completely determine the intersection point of the two satellites.
Now, we proceed to the condition of two way orbits. We proceed as in the previous section but with general orbit parameters.

Assume that the point of intersection occur at point 1 in the first orbit corresponding to a true anomaly, $\varphi_1$, and at point 2 in the second orbit corresponding to a true anomaly, $\varphi_2$. And assume also that the two orbits have common $e$, $i$, and $\omega$. Then it can be shown that the velocity of point 1 relative to the Earth is:

$$\vec{V}_1^R = \begin{cases} 
\sqrt{\frac{\mu}{p_1}} \left( \tau_{11} \tau_{12} + \tau_{13} \tau_{14} \right) + \frac{\omega_E p_1}{1 + e \cos(\varphi_1)} \tau_{15} \\
\sqrt{\frac{\mu}{p_1}} \left( \tau_{21} \tau_{12} + \tau_{23} \tau_{14} \right) - \frac{\omega_E p_1}{1 + e \cos(\varphi_1)} \tau_{25} \\
\sqrt{\frac{\mu}{p_1}} S_i \left( S_{\omega} \tau_{12} + C_{\omega} \tau_{14} \right) \end{cases} \quad (39)$$

where,

$$\tau_{11} = C_{\Omega_1} C_{\omega} - C_i S_{\Omega_1} S_{\omega},$$
$$\tau_{12} = \sin(\varphi_1) \left[ \cos(\varphi_1) (1 - e) - 1 \right],$$
$$\tau_{13} = -C_{\Omega_1} S_{\omega} - C_i S_{\Omega_1} C_{\omega},$$
$$\tau_{14} = 1 - \left[ \cos(\varphi_1) (1 - e) - 1 \right] \cos(\varphi_1),$$
$$\tau_{15} = S_{\Omega_1} \cos(\omega + \varphi_1) + C_{\omega} C_{\Omega_1} \sin(\omega + \varphi_1)$$

$$\tau_{21} = S_{\Omega_1} C_{\omega} + C_{\omega} C_{\Omega_1} S_{\omega},$$
$$\tau_{23} = -S_{\Omega_1} S_{\omega} + C_{\omega} C_{\Omega_1} C_{\omega},$$
$$\tau_{25} = C_{\Omega_1} \cos(\omega + \varphi_1) - C_i S_{\Omega_1} \sin(\omega + \varphi_1)$$

$\vec{V}_2^R$ is defined similar to $\vec{V}_1^R$ with $p_2, \Omega_2$ and $\varphi_2$ replacing $p_1, \Omega_1$ and $\varphi_1$ respectively.

$$\vec{V}_2^R = \begin{cases} 
\sqrt{\frac{\mu}{p_2}} \left( \sigma_{11} \sigma_{12} + \sigma_{13} \sigma_{14} \right) + \frac{\omega_E p_2}{1 + e \cos(\varphi_2)} \sigma_{15} \\
\sqrt{\frac{\mu}{p_2}} \left( \sigma_{21} \sigma_{12} + \sigma_{23} \sigma_{14} \right) - \frac{\omega_E p_2}{1 + e \cos(\varphi_2)} \sigma_{25} \\
\sqrt{\frac{\mu}{p_2}} S_i \left( S_{\omega} \sigma_{12} + C_{\omega} \sigma_{14} \right) \end{cases} \quad (40)$$

where $\sigma_{ij}$ correspond to $\tau_{ij}$.

The condition for having two way orbits is that the vectors $\vec{V}_1^R$, $\vec{V}_2^R$ and the position vector of the intersection point, $\vec{r}_i$, belongs to the same plane. Then we can write this condition as follow:

$$\chi = \vec{r}_i \cdot \left( \vec{V}_1^R \times \vec{V}_2^R \right) = 0 \quad (41)$$

It is difficult to derive analytically an expression that gives the inclination of such orbits, however a numerical solution is developed.
6. Numerical Solution Algorithm

It is possible to introduce a numerical algorithm to find the intersection point of two satellites and the condition of the two orbits such that they constitute a Two-Way Orbit. This is done through two consecutive steps.

6.1. Determine the Intersection Point

Equations (33) and (38) can be solved numerically to find \( \varphi_1 \) and \( \varphi_2 \) as follow:

1. Assume a value for \( \varphi_1 \)
2. Get the corresponding \( \psi_1 \).
3. Given the phasing parameters, \( F_n \) and \( F_d \), From Eq. (38), get \( \psi_2 \).
4. Get the corresponding \( \varphi_2 \).
5. Check if \( \varphi_1 \) and \( \varphi_2 \) satisfy Eq. (33), if not repeat.

This will result in the values of \( \varphi_1 \) and \( \varphi_2 \) at the intersection point given the orbital shape, \( a \), \( e \), and \( \omega \).

6.2. Determine the Orbits Inclination

In this step, we use the Two-Way Orbit condition, Eq. (41), to find the orbit inclination. Given \( \Omega_1 \), \( \Omega_2 \) can calculated as follow:

\[
\Omega_2 = \Omega_1 - 2\pi \frac{F_n}{F_d} \tag{42}
\]

we will then loop on all possible values inclination, and in each time we check if the derived condition is satisfied or not. This will result in all possible values for the inclination, \( i \), completing the five orbital elements. There are many parameters to play with, one case is plotted in Figures 3 and 4 where \( F_n = 1 \) and \( F_d = 4 \). The condition \( \chi = 0 \) satisfaction is investigated and the variation of \( \chi \) with different values for the eccentricity is plotted in Figures 4.

For a Flower Constellation (Mortari, Wilkins, and Bruccoleri, 2004), we have two phasing conditions. The first is used in calculating the intersection point as discussed above. The second is used to calculate \( \Omega_2 \), Eq. (42). So all the calculated satellites constitute a Flower Constellation.
Figure 3. General Two-way orbit inclinations for different values of eccentricities and semi-major axis

Figure 4. General Two-way orbit: The condition X values for different eccentricities

7. Conclusions

In this paper, the concept of Two Way orbits is investigated. The special case of two satellites intersecting at their perigee and apogee locations is solved analytically. The general case of the two satellites intersecting at two general points on their orbits is formulated and solved numerically. The case of two satellites in a Flower Constellation
is investigated and results demonstrated possible existence of general Two-Way orbits.

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References


