Prob. 4.3-11. The solid shaft in Fig. P4.3-11 has a diameter that varies linearly from \( d_0 \) at \( x = 0 \) to \( 2d_0 \) at \( x = L \). It is subjected to a torque \( T_0 \) at \( x = 0 \), and is attached to a rigid wall at \( x = L \). (a) Determine an expression for the maximum (cross-sectional) shear stress in the tapered shaft as a function of the distance \( x \) from the left end. (b) Determine an expression for the total angle of twist of the shaft, \( \phi_0 \). The shear modulus of elasticity is \( G \).

(a) Maximum shear stress

\[
\text{by similar triangles,} \quad \frac{2d_0 - d(x)}{L} = \frac{d(x) - d_0}{x}
\]

\[
d(x) = d(1 + \frac{x}{L})
\]

\[
I_p(x) = \frac{\pi d(x)^4}{32} = \frac{\pi d_0^4}{32} \left(1 + \frac{x}{L}\right)^4
\]

Equilibrium

\( \mathbf{FBD}: \) shaft section, \( 0 < x < L \)

\[
\text{1) } \sum M = 0: T(x) - T_0 = 0 \quad \frac{T(x)}{T_0} = \text{const}
\]

Torsion formula

\[
T(x) = \frac{T_0}{I_p(x)} I_p(x)
\]

\[
T(x) = \frac{T_0}{\pi d_0^3} \left(1 + \frac{x}{L}\right)^{-3}
\]

\[
T_{\text{max}}(x) = \frac{16 T_0}{\pi d_0^3} \left(1 + \frac{x}{L}\right)^{-3}
\]

An. (a)

(b) Total angle of twist

\[
\phi_0 = \phi(L) - \phi(0) = \int_0^L \frac{T(x)}{G I_p(x)} \, dx
\]

\[
= \frac{32 T_0}{\pi G d_0^3} \int_0^L \left(1 + \frac{x}{L}\right)^4 \, dx
\]

\[
= \frac{32 T_0}{\pi G d_0^3} \left[ -\frac{1}{3} \left(1 + \frac{x}{L}\right)^3 \right]_0^L
\]

\[
= \frac{28 T_0 L}{3 \pi G d_0^3}
\]

\[
\phi_0 = \frac{28 T_0 L}{3 \pi G d_0^3}
\]

An. (b)
Prob. 4.3-14. A uniform shaft of diameter $d$, length $L$, and shear modulus $G$, is subjected to a uniformly distributed torque $t_o$ over half of its length, as shown in Fig. P4.3-14. (a) Determine an expression for $r_{max}$ and indicate the location(s) where $r_{max}$ occurs. (b) Determine the angle of rotation at $A, \phi_a$, and the angle of rotation at $B, \phi_B$.

(a) Maximum shear stress

Equilibrium

FBD: shaft section, $\frac{1}{2} < x < L$

\[ T = \frac{t_o}{2} (L-x) \]

Shear stress

\[ (T_{\text{max}}) = I_p \frac{1}{12L} \left( \frac{x}{L} \right)^3 \]  \hspace{1cm} \text{(at } x = \frac{L}{2} \text{)}

\[ (T_{\text{max}})_2 = I_p \frac{8t_o L}{12L} \]  \hspace{1cm} \text{(at } 0 < x < \frac{L}{2} \text{)}

Ans. (a) \[ T_{\text{max}} = \frac{8t_o L}{12L} \]  \hspace{1cm} \text{Note: maximum shear stress occurs at the outer surface of the shaft, for all of member (2) (ie, for } 0 < x \leq L/2 \text{)}

(b) Angles of rotation

Torque - Twist

\[ \phi_1 = \frac{1}{12L} \int_0^L T(x)dx = \frac{32t_o L^4}{12L} \int_0^{L/2} T(x)dx \]

\[ \phi_2 = \frac{T(L)}{12L^2} = \frac{16t_o L}{12L^2} \]

Geometry of Deformation

\[ \phi_a = \phi_1 + \phi_2 \]

\[ \phi_B = \phi_2 \]

Sub. (1) into (2)

\[ \phi_a = \frac{32t_o L^4}{12L} \int_0^{L/2} T(x)dx + \frac{16t_o L}{12L^2} = \frac{4t_o L^2}{12L} + \frac{8t_o L}{12L} = \frac{12t_o L}{12L} \]

\[ \phi_b = \frac{32t_o L^2}{12L} \]

\[ \phi_a = \frac{12t_o L^2}{12L} \]

\[ \phi_B = \frac{8t_o L^2}{12L} \]

Ans. (b)
Prob. 4.3-15. A lunar soil sampler consists of a tubular shaft that has an inside diameter of 45 mm and an outside diameter of 60 mm. The shaft is made of stainless steel, with a shear modulus \( G = 80 \text{ GPa} \). In Fig. 4.3-15, the sampler is being removed from the sampling hole by two 150-N forces that the astronaut exerts at right angles to arm \( DE \) and parallel to the lunar surface. Assume that the external torque that is exerted by the soil from \( A \) to \( B \) is uniformly distributed, with magnitude \( t_b \). (a) Determine the maximum shear stress in the shaft of the sampler tube \( AC \), and indicate the location(s) where it acts. (b) Determine the angle of twist of end \( C \) with respect to end \( A \), that is, determine \( \phi_{CA} = \phi_C - \phi_A \).

\[ \sum M_x = 0 : \]

\[ T_{AB} = 37.5 \text{ N-m} - T_c (l_0 - x) \]  

\[ (T_{\text{max}})_{AB} = \frac{T_{AB} \left( \frac{d_x}{I_p} \right)}{l_0} = \frac{16(37.5 \text{ N-m})(60\text{ mm})}{\pi (60\text{ mm})^2 -(45\text{ mm})^2} = 1.2934 \text{ MPa} \quad (x = 1\text{ m}) \]

\[ (T_{\text{max}})_{BC} = \frac{T_{BC} \left( \frac{d_y}{I_p} \right)}{l_0} = \frac{16(37.5 \text{ N-m})(60\text{ mm})}{\pi (60\text{ mm})^2 -(45\text{ mm})^2} = 1.2934 \text{ MPa} \quad (1\text{ m} < x < 2.5\text{ m}) \]

**Ans.** (a) \( T_{\text{max}} = 1.293 \text{ MPa} \) 

**Note:** Maximum shear stress occurs at the outer surface of member (2) (i.e., \( 1 \text{ m} < x < 2.5 \text{ m} \))

b) Relative angle of twist

**Torque-Twist**

\[ \phi_{AB} = \int_0^{l_0} \frac{T(x)}{G I_p} \, dx = \frac{32 T_{BC} L_{BC}}{G I_p} \int_0^{l_0} T(x) \, dx \]

\[ \phi_{BC} = \frac{TL_{BC}}{G I_p} = \frac{32 T_{BC} L_{BC}}{G I_p} \]

**Geometry of Deformation**

\[ \phi_{AB} = \phi_B - \phi_A \quad \phi_{BC} = \phi_C - \phi_B \quad \Rightarrow \phi_{CA} = \phi_C - \phi_A = \phi_{AB} + \phi_{BC} \]

(continued on next page)
\[ \Phi_{c/A} = 1.078 \times 10^{-3} \text{ rad} \]  
\[ 
\text{Ans. (b)}
\]
*Prob. 4.3-16. A uniform shaft of diameter \( d \) and length \( L \) is fixed to rigid "walls" at ends \( A \) and \( B \) and is subjected to a quadratically varying distributed external torsional loading (Note that Fig. P4.3-16 shows reaction torques \( T_A \) and \( T_B \) and this external torsional loading \( t(x) \) all acting in the positive right-hand-rule sense with respect to the \( x \) axis.)

By using a free-body diagram of the entire shaft, together with the two fixed-end constraint condition, \( \phi(0) = \phi(L) = 0 \), determine expressions for the reaction torques \( T_A \) and \( T_B \) at the fixed ends. Assume that linearly elastic behavior results from the torsional loading \( t(x) \). (Note that this is a statically indeterminate torsion member since there are two unknown reaction torques but only one equilibrium equation.)

**Equilibrium**

1. \[ EBD: \text{shaft section, } 0 < x < L \]
2. \[ \sum M_x = 0: \quad T(x) = -T_A - \int_0^x t(\xi) \, d\xi = -T_A - E_c \left[ x - \frac{x^3}{3L^2} \right] \]

**Torque-Twist**

1. \[ \phi = \int_0^L C \frac{t}{r} \, dx = \frac{3E}{12} \int_0^L T(x) \, dx \]

**Geometry of Deformation**

1. \[ \phi_A = \phi_B = 0 \]
2. \[ \phi_B = 0 \]

Substituting (1) \( \rightarrow \) (2) \( \rightarrow \) (3)

\[ \phi = \frac{3E}{12} \int_0^L \left[ -T_A - t(x) \left( x - \frac{x^3}{3L^2} \right) \right] \, dx = 0 \]

\[ \left[ -T_A x - t_0 \left( x - \frac{x^3}{3L^2} \right) \right]_0^L = 0 \]

\[ -T_A L - \frac{5t_0 L^2}{12} = 0 \]

\[ T_A = \frac{-5t_0 L}{12} \]

\[ T_B = \frac{-T_A - 2t_0 L}{3} = \frac{-t_0 L}{4} \]

**Conclusion:**

\[ T_A = \frac{-5t_0 L}{12} \]
\[ T_B = \frac{-t_0 L}{4} \]
Problems 4.9-1 through 4.9-7. For the thin-wall tubular sections shown in Figs. P4.9-1 through P4.9-7; (a) determine the maximum shear stress in the cross section, and (b) determine the value of the torsion constant, \( J \).

Prob. 4.9-1. See the problem statement above, and use \( T = 500 \text{ kip} \cdot \text{in.} \)

Solution:

(a) Maximum shear stress,

\[ \tau_{\text{max}} = \frac{T}{2 \pi r t} = \frac{500 \text{ kip} \cdot \text{in.}}{2 \pi (0.25 \text{ in.}) (10 \text{ in.})} = 5.00 \text{ ksi} \]

\[ \tau_{\text{max}} = 5.00 \text{ ksi} \]

(b) Torsion constant.

\[ J = \frac{4 a_m^2}{\int_{z=0}^{t/2} \frac{dz}{z}} = \frac{4 (120 \text{ in.})^2}{\int_{z=0}^{t/2} \frac{dz}{z}} = \frac{4 (120 \text{ in.})^2}{\left[ 2 (z \text{ in.}) + 2 (10 \text{ in.}) \right]} = 666.7 \text{ in}^4 \]

\[ J = 666.7 \text{ in}^4 \]

Prob. 4.9-2. See the problem statement preceding Prob. 4.9-1, and use \( T = 5 \text{ kN} \cdot \text{m} \).

(a) Maximum shear stress,

\[ a_m = 160 \text{ mm} (120 \text{ mm}) = 19200 \text{ mm}^2 \]

\[ \tau_{\text{max}} = \frac{T}{2 t \min a_m} = \frac{T}{2 t \min a_m} = \frac{5 \text{ kN} \cdot \text{m}}{2 (160 \text{ mm})(19200 \text{ mm}^2)} = 21.701 \text{ MPa} \]

\[ \tau_{\text{max}} = 21.7 \text{ MPa} \]

(b) Torsion constant,

\[ \frac{d^2 a_m}{dz^2} = \frac{z}{t_i} = \frac{2 \left[ 160 \text{ mm} + 120 \text{ mm} \right]}{16 \text{ mm}} = 93.333 \]

\[ J = \frac{4 (19200 \text{ mm}^2)^2}{93.333} = 1.580 \times 10^9 \text{ m}^4 \]

\[ J = 1.580 \times 10^9 \text{ m}^4 \]
Prob. 4.9-3. See the problem statement preceding Prob. 4.9-1, and use $T = 5 \text{kN} \cdot \text{m}$.

Solution:

(a) Maximum shear stress,
\[
A_m = \frac{(120 \text{mm})(60 \text{mm}) + \pi (30 \text{mm})^2}{2} = 1,002.74 \times 10^4 \text{mm}^2
\]
\[
T_{\text{max}} = \frac{T}{2\pi A_m} = \frac{5 \text{kN} \cdot \text{m}}{2 \pi (9.5 \text{mm})(1,002.74 \times 10^4 \text{mm}^2)} = 26.24 \text{ MPa}
\]

\[
T_{\text{max}} = 26.2 \text{ MPa}
\]

(b) Torsion constant,
\[
\int \frac{ds}{t_i} = \sum \frac{S_i}{t_i} = \frac{2(120 \text{ mm}) + 20(30 \text{ mm})}{9.5 \text{ mm}} = 45.105
\]
\[
J = \frac{4A_m^2}{\int \frac{ds}{t_i}} = \frac{4(1,002.74 \times 10^4 \text{mm}^2)^2}{45.105} = 8.92 \times 10^6 \text{mm}^4
\]

Prob. 4.9-4. See the problem statement preceding Prob. 4.9-1, and use $T = 800 \text{ kip} \cdot \text{in}$.

(a) Maximum shear stress,
\[
A_m = \frac{18 \text{ in.}(12 \text{ in}) + \frac{\pi}{2}(6 \text{ in})^2}{2} = 272.55 \text{ in}^2
\]
\[
T_{\text{max}} = \frac{T_{\text{max}}}{2\pi A_m} = \frac{T}{2\pi A_m} = \frac{800 \text{ kip} \cdot \text{in}}{2(0.5 \text{ in})(272.55 \text{ in}^2)} = 2,935.3 \text{ ksi}
\]

\[
T_{\text{max}} = 2,944 \text{ ksi}
\]

(b) Torsion constant,
\[
\int \frac{ds}{t_i} = \sum \frac{S_i}{t_i} = \frac{2(18 \text{ in}) + 12 \text{ in} + \pi(6 \text{ in})}{0.5 \text{ in.}} = 133.70
\]
\[
J = \frac{4A_m^2}{\int \frac{ds}{t_i}} = \frac{4(272.55 \text{ in}^2)^2}{133.70} = 2,222.4 \text{ in}^4
\]

\[
J = 2,220 \text{ in}^4
\]
Prob. 4.9-3. See the problem statement preceding Prob. 4.9-1, and use $T = 5 \text{ kN} \cdot \text{m}$.

**Solution:**

(a) Maximum shear stress.

$$A_m = (120 \text{ mm})(60 \text{ mm}) + \pi (30 \text{ mm})^2 = 1.00274 \times 10^4 \text{ mm}^2$$

$$T_{\max} = \frac{T}{2bA_m} = \frac{5 \text{ kN} \cdot \text{m}}{2(9.5 \text{ mm})(1.00274 \times 10^4 \text{ mm}^2)} = 26.2 \text{ MPa}$$

$$T_{\max} = 26.2 \text{ MPa}$$

(b) Torsion constant.

$$\int \frac{ds}{t(\theta)} = \sum \frac{s_i t_i}{t} = \frac{2(120 \text{ mm}) + 20(30 \text{ mm})}{9.5 \text{ mm}} = 45.105$$

$$J = \frac{4A_m^3}{\int \frac{ds}{t(\theta)}} = \frac{4(1.00274 \times 10^4 \text{ mm}^2)^3}{45.105} = 8.917(10^6) \text{ mm}^4$$

$$J = 8.917(10^6) \text{ mm}^4$$

Prob. 4.9-4. See the problem statement preceding Prob. 4.9-1, and use $T = 800 \text{ kip} \cdot \text{in}$.

(a) Maximum shear stress.

$$A_m = 18 \text{ in.(12 in.)} + \frac{\pi}{2}(6 \text{ in.})^2 = 272.55 \text{ in}^2$$

$$T_{\max} = \frac{T_{\min}A_m}{2T_{\max}} = \frac{T_{\max}}{2(0.5 \text{ in})(272.55 \text{ in}^2)} = 2.9353 \text{ kip}$$

$$T_{\max} = 2.93 \text{ kip}$$

(b) Torsion constant.

$$\int \frac{ds}{t(\theta)} = \sum \frac{s_i t_i}{t} = \frac{2(18 \text{ in}) + 12 \text{ in.} + \pi(6 \text{ in.})}{0.5 \text{ in.}} = 133.70$$

$$J = \int \frac{ds}{t(\theta)} = \frac{4(272.55 \text{ in}^2)^2}{133.70} = 2222.4 \text{ in}^4$$

$$J = 2222.4 \text{ in}^4$$
Prob. 4.9-9. The torque tube in Fig. P4.9-9 has an elliptical cross section. Express the shear stress on the cross section in terms of the following parameters: \( T, a, b, \) and \( t. \) (See the Table of Geometric Properties of Plane areas inside the back cover.)

**Solution:**

\[
\tau_m = \frac{T}{2\pi a b t}
\]

\[
\tau = \frac{T}{2\pi a b t}
\]

Prob. 4.9-10. A tubular shaft having an inside diameter of \( d_i = 2.0 \text{ in.} \) and a wall thickness of \( t = 0.20 \text{ in.} \), is subjected to a torque \( T = 5 \text{ kip} \cdot \text{in.} \) (Note: The wall thickness is one-tenth of the inner diameter so that you can examine the range of validity of thin-wall torsion theory.) Determine the maximum shear stress in the tube: (a) using the exact theory for torsion of circular shafts, and (b) using the (approximate) thin-wall torsion theory of this section.

**Exact Theory**

\[
r_o = \frac{d_o}{2} = \frac{d_i + 2t}{2} = \frac{2.0\text{in} + 2(0.20\text{in})}{2} = 1.2 \text{ in.}
\]

\[
I_p = \frac{\pi}{2} \left( \frac{d_i}{2} \right)^4 - \frac{\pi}{2} \left( \frac{d_i + 2t}{2} \right)^4 = \frac{\pi}{2} \left( \frac{2.0\text{in}}{2} \right)^4 - \frac{\pi}{2} \left( \frac{2.4\text{in}}{2} \right)^4 = 1.68 \text{ in.}^4
\]

\[
\tau_{max} = \frac{T}{I_p} = \frac{5 \text{ kip} \cdot \text{in.}}{1.68 \text{ in.}^4} = 3.56 \text{ ksi}
\]

**Thin-Wall Theory**

\[
\tau_{max} = \frac{T}{2Em \cdot \frac{\pi}{2} \left( \frac{d_i}{2} \right)^4} = \frac{T}{5 \text{ kip} \cdot \text{in.}} = 3.90 \text{ in.}^2
\]

\[
\tau_{max} = \frac{T}{2Em \cdot \frac{\pi}{2} \left( \frac{d_i}{2} \right)^4} = \frac{T}{5 \text{ kip} \cdot \text{in.}} = 3.29 \text{ ksi}
\]
Prob. 4.9-12. A 4-in.-square steel tube has a wall thickness \( t = 0.25 \text{ in.} \). You are to compare the torsion behavior of the square tube to that of a tube with circular cross section having the same median-curve length \( (L_m = 16 \text{ in.}) \). Use thin-wall torsion theory to (a) determine the shear-stress ratio \( \tau_s/\tau_c \), where \( \tau_s \) and \( \tau_c \) are the maximum shear stresses in the circular tube and the square tube, respectively, when both members are subjected to a torque \( T = 80 \text{ kip \cdot in.} \). (b) Determine the ratio \( J/J_s \), where \( J_s \) and \( J_c \) are the respective circle and square area properties defined by the torsional rigidity equation, Eq. 4.34.

(a) maximum shear stress ratio

\[
\left( \frac{a_m}{a_m} \right)_c = \left( \frac{4 \text{ in.}}{2 \text{ in.}} \right)^2 = \frac{16 \text{ in.}^2}{4 \text{ in.}^2} = 4
\]

\[
(L_m)_s = (L_m)_c = \frac{T d}{\tau_s} = \frac{T d}{4 \text{ in.}^2} = \frac{16 \text{ in.}^2}{2 \text{ in.}^2} = 8 \text{ in.}
\]

\[
(a_m)_c = \frac{T d}{4 \text{ in.}^2} = \frac{16 \text{ in.}^2}{2 \text{ in.}^2} = 8 \text{ in.}
\]

\[
\frac{T_c}{T_s} = \frac{(a_m)_c}{(a_m)_s} = \frac{16 \text{ in.}^2}{20.372 \text{ in.}^2} = 0.785
\]

(b) torsion constant ratio

\[
\frac{J_c}{J_s} = \frac{4 \left( a_m \right)_s^2}{L_s} = \frac{4 \left( 16 \text{ in.}^2 \right)^2}{16 \text{ in.}} = 16 \text{ in.}^4
\]

\[
J_s = \int_{cm_s} \frac{dA}{t(s)} = \int_{cm_s} \frac{dA}{16 \text{ in.}} = 16 \text{ in.}^4
\]

\[
J_c = \int_{cm_c} \frac{dA}{t(s)} = \int_{cm_c} \frac{dA}{8 \text{ in.}} = 25.938 \text{ in.}^4
\]

\[
\frac{J_c}{J_s} = \frac{25.938 \text{ in.}^4}{16 \text{ in.}^4} = 1.621
\]
*Prob. 4.9-13. A thin-wall tube has uniform thickness \( t \) and cross-sectional dimensions \( b \) and \( h \), measured to the median line of the cross section. Let the length of the median curve, \( L_m = 2b + 2h \), and the thickness \( t \) be constant (hence, the weight will be constant), but let the ratio \( \alpha = \frac{b}{h} \) vary. (a) Determine an expression that relates the maximum shear stress in a rectangular tube to the ratio \( \alpha \). (b) From your result in (a), show the maximum shear stress will be smallest when the tube is square (i.e., when \( \alpha = 1 \)).

\[ a) \text{ maximum shear stress} \]
\[ A_m = bh = \alpha h^2 \]
\[ L_m = 2(b + h) = 2(\alpha + 1)h \]
\[ h = \frac{2(\alpha + 1)}{L_m} \]
\[ \tau_{\text{max}} = \frac{T_{\text{max}}}{2tA_m} = \frac{T}{2t\alpha h^2} = \frac{2T(\alpha + 1)^2}{\alpha L_m} \]
\[ \tau_{\text{max}}(\alpha) = \frac{2T(\alpha + 1)^2}{\alpha L_m} \]

\[ b) \text{ minimization} \]
\[ \frac{d\tau_{\text{max}}(\alpha)}{d\alpha} = \frac{2T}{\alpha L_m^2} \left[ \frac{2(\alpha + 1)^2(\alpha - (\alpha + 1)^2)}{\alpha^3} \right] = 0 \]
\[ 2\alpha - \alpha - 1 = 0 \]
\[ \alpha - 1 = 0 \]
\[ \alpha = 1 \]

\[ \Rightarrow \text{For } \alpha = 1, \tau_{\text{max}} \text{ is at a minimum.} \]