A. Symbols: Rod Along the z-axis

\[ m_t(z) = \text{distributed torque load which can be a function of } z \quad \text{[force*length/length]} \]
\[ M_t(z) = \text{resultant torque which can be a function of } z \quad \text{[force*length]} \]
\[ J(z) = \text{polar moment of inertia which can be a function of } z \quad \text{[length^4]} \]
\[ \sigma_{z\theta}(z) = \text{shear stress which can be a function of } z \quad \text{[force/length^2]} \]
\[ \varepsilon_{z\theta}(z) = \text{shear strain which can be a function of } z \quad \text{[length/length]} \]
\[ u_{\theta}(z) = \text{displacement } (\theta \text{-direction}) \text{ which can be a function of } z \quad \text{[length]} \]
\[ G(z) = \text{Shear Modulus (material property for elasticity)} \quad \text{[force/length^2]} \]
\[ \phi(z) = \text{angle of twist about the z-axis} \quad \text{[length/length]} \]
\[ R = \text{radius of rod} \quad \text{[length]} \]
\[ D = \text{diameter of rod} \quad \text{[length]} \]

B. Deriving the governing differential equation (GDE)

1. Assumptions:
   1. Rod = L > w or h with circular cross-section
   2. Static (body is at rest)
   3. T = constant (i.e. isothermal)
   4. \( \sigma_{z\theta} \neq 0 \) all other stresses \( \approx 0 \)
   5. \( \sigma_{z\theta} = \sigma_{z\theta}(z) \)
   6. Isotropic Linear elastic material behavior (i.e. Hookean)
   7. Smooth A(x)
   8. Load evenly distributed around cross-section (or along centroidal axis)
   9. \( u_r = u_z \approx 0 \)
   10. Small displacements gradients
   11. \( u_\theta = r \phi \)
   12. \( \phi = \phi(z) \)
   13. Gravity is negligible

2. Field Equations

   Equilibrium \( \frac{dM_t(z)}{dz} + m_t(z) = 0 \) \quad \text{(EQ#1) (i.e. C.O.A.M.)}

   Resultant \( M_t(z) = \int_A r \sigma_{z\theta} dA \quad \text{also... } J = \int_0^{2\pi} \int_0^L r^2 dr d\theta \) \quad \text{(EQ#2) (a definition)}

   Constitutive \( \sigma_{z\theta} = 2G \varepsilon_{z\theta} \) \quad \text{(EQ#3) (i.e. linear elasticity (Hookean) or stress-strain relations)}

   Kinematic \( \varepsilon_{z\theta} = \frac{1}{2} r \frac{d\phi}{dz} \) \quad \text{(EQ#4) (i.e. strain/displacement relations for small displacements)}

3. Unknowns: \( M_t(z), \sigma_{z\theta}(z), \varepsilon_{z\theta}(z), \phi(z) = 4 \) (total of 4 unknowns vs. 4 equations \( \Rightarrow \) tractable)
4. Derivation of GDE:

\[(\text{EQ#4})\] into \[(\text{EQ#3})\] gives:
\[
\sigma_{rz} = 2G \frac{1}{2} r \frac{d\phi}{dz} \Rightarrow \sigma_{rz} = Gr \frac{d\phi}{dz} \quad \text{(EQ#5)}
\]

\[(\text{EQ#5})\] into \[(\text{EQ#2})\] gives:
\[
M_i(z) = GJ \frac{d\phi}{dz} \quad \text{where:} \quad J \equiv \frac{\pi D^4}{32} \quad \text{for a circle} \quad \text{(EQ#6)}
\]

(Used for torque B.C.’s just like just like similar equation for uniaxial bars)

\[(\text{EQ#6})\] into \[(\text{EQ#1})\] gives:
\[
\frac{d}{dz} \left[ GJ \frac{d\phi}{dz} \right] + m_i(z) = 0 \quad \text{(EQ#7)}
\]

5. B.C.’s Need 2 (one at each end)
- @ \(z = 0\) either \(M_i(z = 0)\) or \(\phi(z = 0)\) are known
- @ \(z = L\) either \(M_i(z = L)\) or \(\phi(z = L)\) are known

C. Simplified Case similar to “FLEA” for uniaxial bars

1. Specified geometry and loading (THIS DOES NOT WORK OTHERWISE!)

2. Assumptions
   1. all of the above assumptions
   2. one end is fixed
   3. No distributed load, i.e. \(m_i = 0\)
   4. Concentrated end torque ONLY of \(T_L\)
   5. \(G, J\) are constant throughout the entire length of the bar, i.e. the material is spatially homogeneous and the cross section is prismatic

\[
\begin{array}{c}
\begin{array}{c}
\text{\(T_L\)}
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\]

3. Solving governing differential equation (GDE) for this particular problem gives:
\[
\phi(z) = \frac{T_Lz}{GJ}
\]

or evaluating for the rotation at the end, \(z = L\), gives: \(\phi(z = L) = \frac{T_L L}{GJ}\)